Spatial Transforms

COMP 6912: Autonomous Robotics

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Introduction

- Spatial transforms describe coordinate frames (a.k.a reference frames, coordinate systems, or often just *frames*)
- We have already seen two different reference frames:
  - Inertial (or global) reference: Defined by origin O and axes X_I and Y_I
  - Robot reference frame: Defined **w.r.t.** the inertial frame by origin P and axes X_R and Y_R
  - Both were specialized to motion in the plane, we need a more general representation for motion in 3-D
- We will look at transforms in two different ways: as mappings and as operators
- We will discuss two different orientation representations: rotation matrices and quaternions

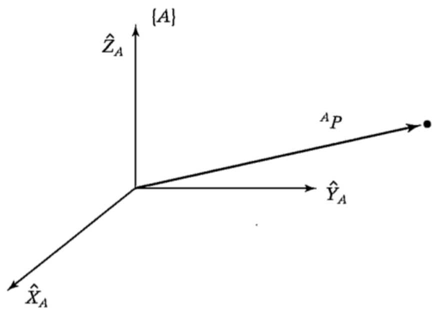
Notation

We use the notation from "Introduction to Robotics: Mechanics and Control", 3rd Edition by John J. Craig
Point P described in frame A is denoted as follows:

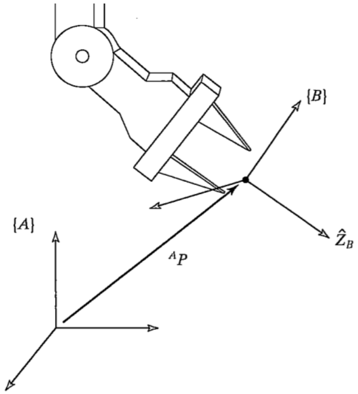
$$^{A}P = \begin{bmatrix} p_x \\ p_y \\ p_z \end{bmatrix}$$

The A superscript appears to the left of the actual point and means that we are describing the point with respect to A. The same point described in frame B would be ^{B}P.

Frame A is described by three mutually orthogonal unit vectors $\hat{X}_A$, $\hat{Y}_A$, and $\hat{Z}_A$. The components of ^{A}P are the projections of the vector corresponding to this point onto these axes.



Lets now say that this point ${}^AP$ actually gives the origin of another frame called B. Perhaps this frame gives the position of a robot's end effector.



We can describe one frame w.r.t. to another by specifying the **rotation** and **translation** of the movement that separates them.

Rotation

The **rotation** of frame B w.r.t. frame A is defined through the mutually orthogonal unit vectors ${}^A\hat{X}_B$, ${}^A\hat{Y}_B$, and ${}^A\hat{Z}_B$. If we stack these three vectors as columns of a 3×3 matrix we get the rotation matrix that encodes the orientation of frame B w.r.t. frame A:

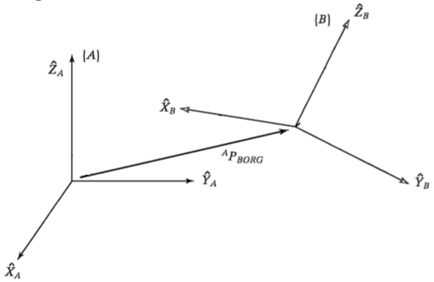
$$ {}^A_BR = \left[\; {}^A\hat{X}_B \quad {}^A\hat{Y}_B \quad {}^A\hat{Z}_B \; \right] = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix} $$

For justification, see Craig's book, chapter 2. You see that the rotation matrix is special: it has mutually orthogonal columns. It turns out that the rows are mutually orthogonal too. Also, the inverse of the rotation is just equal to the transpose.

$$ {}^A_BR = {}^B_AR^{-1} = {}^B_AR^T $$

Translation

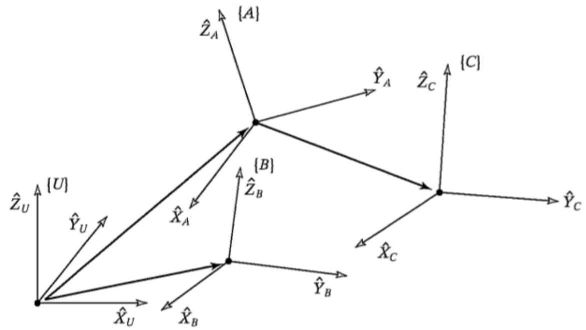
We describe the **translation** between frames by specifying the origin of B w.r.t. A as ${}^AP_{BORG}$.



We now have everything we need to describe frame B:

$$ \{B\} = \left\{ {}^A_BR, {}^AP_{BORG} \right\} $$

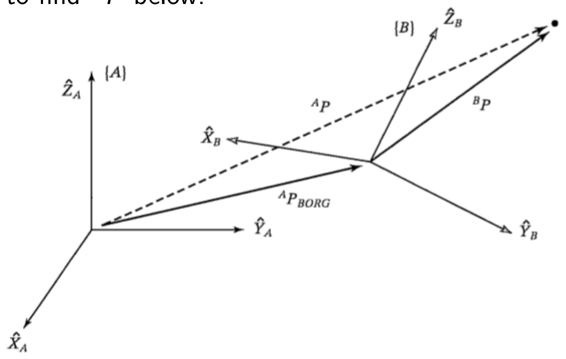
There is usually a universal (a.k.a. global or inertial) frame that others are defined w.r.t.. However, its often the case that we don't have a direct description of a frame with respect to the universal (e.g. for frame C below).

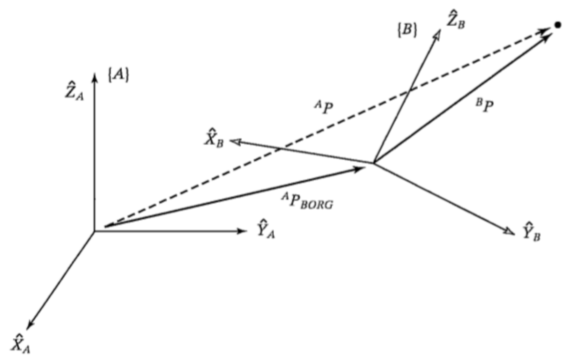


We would like a way of mapping descriptions of points from one frame to another.

Mappings

A mapping is a change in our description of the same entity from one frame to another. Lets say we know the position of a point w.r.t. B but we want to compute it w.r.t. A. For example, we wish to find ${}^A P$ below:





If we have ${}^B P$ and our description of B we can get ${}^A P$.

$$ {}^A P = {}^A_B R \, {}^B P + {}^A P_{BORG} $$

Here again is this mapping:

$$ {}^A P = {}^A_B R \, {}^B P + {}^A P_{BORG} $$

It would be convenient to combine both operations together into one big matrix so that we can do the following:

$$ {}^A P = {}^A_B T \, {}^B P $$

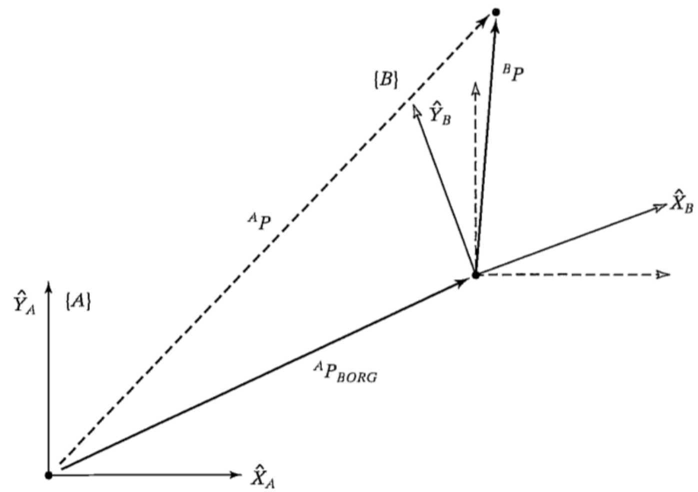
We can do this by moving to **homogeneous coordinates**. In homogeneous coordinates, we just augment our 3×1 vectors with an additional row that is always set to 1.

$$ \begin{bmatrix} {}^A P \\ 1 \end{bmatrix} = \left[\begin{array}{ccc|c} & {}^A_B R & & {}^A P_{BORG} \\ \hline 0 & 0 & 0 & 1 \end{array} \right] \begin{bmatrix} {}^B P \\ 1 \end{bmatrix} $$

This yields the equation at the top of this slide, as well as $1 = 1$ (uninformative, but reassuring). We can represent other sorts of transformations (e.g. shearing, scaling, perspective projection) by modifying particular entries in the augmented matrix.

2-D Example

Frame B lies at position (10, 5) w.r.t. frame A and is rotated by 30^o. Given ${}^B P = [3\ 7\ 0]^T$ find ${}^A P$.



We get the following transform matrix:

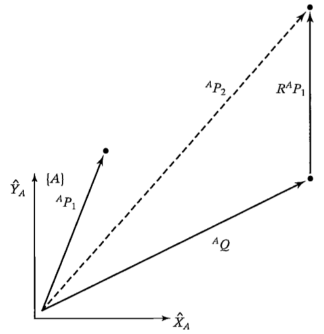
$$ {}^A_BT = \begin{bmatrix} 0.866 & -0.500 & 0.000 & 10.0 \\ 0.500 & 0.866 & 0.000 & 5.0 \\ 0.000 & 0.000 & 1.000 & 0.0 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

We now apply the transform (notice how the B's "cancel"):

$$ \begin{aligned} {}^AP &= {}^A_BT\,{}^BP \\ &= {}^A_BT \begin{bmatrix} 3 \\ 7 \\ 0 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} 9.098 \\ 12.562 \\ 0 \\ 1 \end{bmatrix} \end{aligned} $$

Transform as Operator

We have used transforms to relate the mapping of a point from one frame to another. We can also view a transform as an **operator** which translates and/or rotates points. For example, lets say we have a point ${}^AP_1 = [3\ 7\ 0]^T$ and we wish to rotate it by 30^o about $\hat{Z}$ and translate it by $[10\ 5\ 0]^T$. Now we wish to find the transformed point ${}^AP_2$, pictured below:

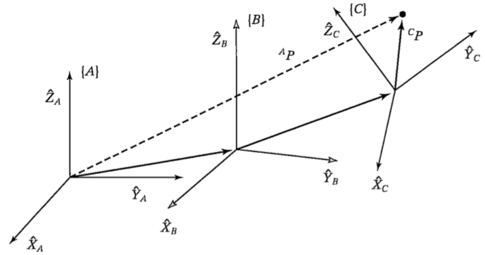


This example is identical to the previous one, except that we view this as an operator, not as a mapping. In this case, there is only one frame, A. The transformation matrix is the same as before, although we have no preceding super- or subscripts because the operation takes place within the same frame.

$$ T = \begin{bmatrix} 0.866 & -0.500 & 0.000 & 10.0 \\ 0.500 & 0.866 & 0.000 & 5.0 \\ 0.000 & 0.000 & 1.000 & 0.0 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

$$ \begin{aligned} {}^AP_2 &= T\,{}^AP_1 \\ &= T \begin{bmatrix} 3 \\ 7 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 9.098 \\ 12.562 \\ 0 \\ 1 \end{bmatrix} \end{aligned} $$

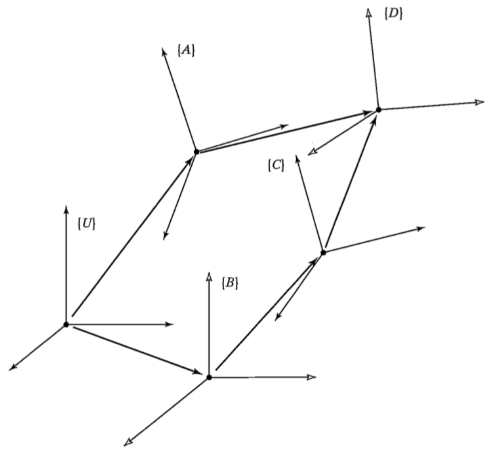
Compound Transforms



Considering the above with ${}^CP$ given, we can express this point in frames B and then A,

$$ \begin{aligned} {}^BP &= {}^B_CT\,{}^CP \\ {}^AP &= {}^A_BT\,{}^BP \\ &= {}^A_BT\,{}^B_CT\,{}^CP \end{aligned} $$

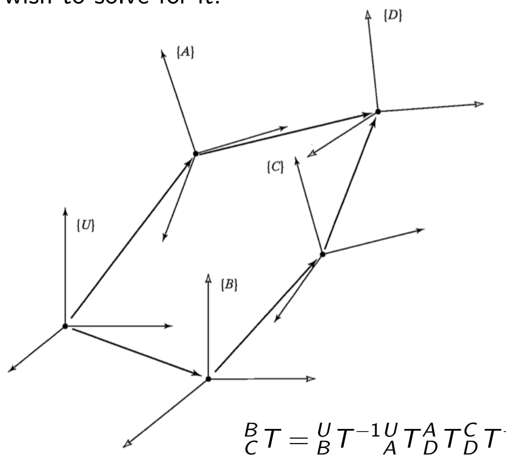
We can define a direct $C \rightarrow A$ transform: ${}^A_CT = {}^A_BT\,{}^B_CT$.



We can express the transform of D with respect to U in two different ways:

$$ {}^U_D T = {}^U_A T\, {}^A_D T \quad \text{and} \quad {}^U_D T = {}^U_B T\, {}^B_C T\, {}^C_D T $$

The direction of the arrows indicates how existing transforms are defined. Lets say that all transforms but ${}^B_C T$ are known and we wish to solve for it.



$$ {}^B_C T = {}^U_B T^{-1}\, {}^U_A T\, {}^A_D T\, {}^C_D T^{-1} $$

Representing 3-D Orientation

Rotation matrices must be orthonormal, meaning that their rows and columns are orthogonal unit vectors. Even further, rotation matrices must have a determinant of +1. Cayley's formula for orthonormal matrices states that any such matrix can be rewritten as follows:

$$ R = (I - S)^{-1}(I + S) $$

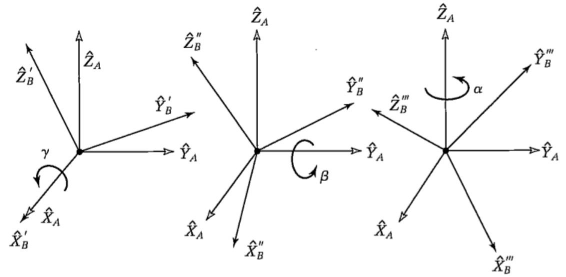
where

$$ S = \begin{bmatrix} 0 & -s_z & s_y \\ s_z & 0 & -s_x \\ -s_y & s_x & 0 \end{bmatrix} $$

The important point is that a rotation matrix actually has only three free parameters. In fact any 3-D rotation can be represented by as little as three numbers. We will look at a couple of example representations...

X-Y-Z Fixed Angles

We can describe the rotation of a frame B with respect to a frame A as a sequence of rotations about $\hat{X}_A$, $\hat{Y}_A$, and $\hat{Z}_A$ (other orders beside X-Y-Z are possible). We are rotating about A which is fixed.

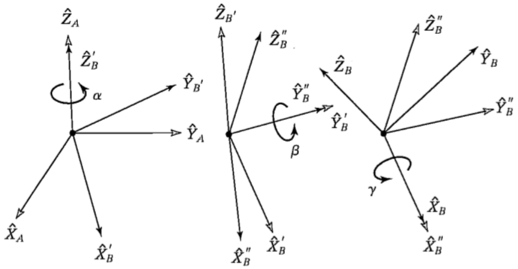


The rotation angles are γ, β, and α. We can generate a rotation matrix by combining all three:

$$ {}^A_B R_{XYZ}(\gamma, \beta, \alpha) = R_Z(\alpha) R_Y(\beta) R_X(\gamma) $$
$$ = \begin{bmatrix} c\alpha & -s\alpha & 0 \\ s\alpha & c\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c\beta & 0 & s\beta \\ 0 & 1 & 0 \\ -s\beta & 0 & c\beta \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & c\gamma & -s\gamma \\ 0 & s\gamma & c\gamma \end{bmatrix} $$

Euler Angles

Euler angles are defined by rotating the "moving" frame, B, about its own axes. The following is for the order Z-Y-X:



We can view this as a series of rotations: $A \to B'$, $B' \to B''$, and $B'' \to B$, where B' and B'' are intermediary frames.

$$ {}^{A}_{B}R = {}^{A}_{B'}R \, {}^{B'}_{B''}R \, {}^{B''}_{B}R $$

From this perspective the rotation by γ happens first, then β, then α. Interestingly, this is the same order and the same angles as for the X-Y-Z fixed angle representation—the two are equivalent!

Various Conventions

- So the X-Y-Z fixed angle convention is equivalent to the Z-Y-X Euler angle convention. There are 12 equivalent pairs of conventions.
- We also have the **axis-angle** representation which represents the rotation via an equivalent axis of rotation specified by a unit vector $\hat{N}$ (2 free parameters) and an angle θ.
- Finally, the unit quaternion representation (see additional notes) is cleanly defined from the axis-angle representation:

$$ q = \cos\frac{\theta}{2} + \sin\frac{\theta}{2}\hat{N} $$

- Note the difference between the representation of a rotation and a rotation operator. The only rotation operators we have seen our rotation matrices and unit quaternions.