Spatial transforms describe coordinate frames (a.k.a reference frames, coordinate systems, or often just *frames*)

We have already seen two different reference frames:

- Inertial (or global) reference: Defined by origin $O$ and axes $X_I$ and $Y_I$
- Robot reference frame: Defined w.r.t. the inertial frame by origin $P$ and axes $X_R$ and $Y_R$

Both were specialized to motion in the plane, we need a more general representation for motion in 3-D

We will look at transforms in two different ways: as mappings and as operators

We will discuss two different orientation representations: rotation matrices and quaternions
We use the notation from "Introduction to Robotics: Mechanics and Control", 3rd Edition by John J. Craig

Point $P$ described in frame $A$ is denoted as follows:

$$A^P = \begin{bmatrix} p_x \\ p_y \\ p_z \end{bmatrix}$$

The $A$ superscript appears to the left of the actual point and means that we are describing the point with respect to $A$. The same point described in frame $B$ would be $B^P$. 
Frame $A$ is described by three mutually orthogonal unit vectors $\hat{X}_A$, $\hat{Y}_A$, and $\hat{Z}_A$. The components of $^A P$ are the projections of the vector corresponding to this point onto these axes.
Lets now say that this point $A_P$ actually gives the origin of another frame called $B$. Perhaps this frame gives the position of a robot’s end effector.

We can describe one frame w.r.t. to another by specifying the rotation and translation of the movement that separates them.
The rotation of frame $B$ w.r.t. frame $A$ is defined through the mutually orthogonal unit vectors $\hat{X}_B$, $\hat{Y}_B$, and $\hat{Z}_B$. If we stack these three vectors as columns of a $3 \times 3$ matrix we get the rotation matrix that encodes the orientation of frame $B$ w.r.t. frame $A$:

$$\begin{align*}
A_B R &= \begin{bmatrix}
\hat{X}_B \\
\hat{Y}_B \\
\hat{Z}_B
\end{bmatrix} =
\begin{bmatrix}
r_{11} & r_{12} & r_{13} \\
r_{21} & r_{22} & r_{23} \\
r_{31} & r_{32} & r_{33}
\end{bmatrix}
\end{align*}$$

For justification, see Craig’s book, chapter 2. You see that the rotation matrix is special: it has mutually orthogonal columns. It turns out that the rows are mutually orthogonal too. Also, the inverse of the rotation is just equal to the transpose.

$$A_B R = B_A R^{-1} = B_A R^T$$
We describe the translation between frames by specifying the origin of $B$ w.r.t. $A$ as $^A P_{BORG}$.

We now have everything we need to describe frame $B$:

$$\{B\} = \left\{^A P_{BORG}, ^A R_B \right\}$$
There is usually a universal (a.k.a. global or inertial) frame that others are defined w.r.t.. However, its often the case that we don’t have a direct description of a frame with respect to the universal (e.g. for frame C below).

We would like a way of mapping descriptions of points from one frame to another.
A mapping is a change in our description of the same entity from one frame to another. Let's say we know the position of a point w.r.t. \( B \) but we want to compute it w.r.t. \( A \). For example, we wish to find \( ^A P \) below:
If we have $^B P$ and our description of $B$ we can get $^A P$.

$$^A P = {^A B}R^B P + {^A P}_{BORG}$$
Here again is this mapping:

\[ A P = \frac{A}{B} R^B P + A P_{BORG} \]

It would be convenient to combine both operations together into one big matrix so that we can do the following:

\[ A P = \frac{A}{B} T^B P \]

We can do this by moving to **homogeneous coordinates**. In homogeneous coordinates, we just augment our \(3 \times 1\) vectors with an additional row that is always set to 1.

\[
\begin{bmatrix}
A P \\
1
\end{bmatrix} = 
\begin{bmatrix}
\frac{A}{B} R & A P_{BORG} \\
0 & 0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
B P \\
1
\end{bmatrix}
\]

This yields the equation at the top of this slide, as well as \(1 = 1\) (uninformative, but reassuring). We can represent other sorts of transformations (e.g. shearing, scaling, perspective projection) by modifying particular entries in the augmented matrix.
Frame $B$ lies at position $(10, 5)$ w.r.t. frame $A$ and is rotated by $30^\circ$. Given $^B P = [3 \ 7 \ 0]^T$ find $^A P$. 
We get the following transform matrix:

\[
\begin{bmatrix}
0.866 & -0.500 & 0.000 & 10.0 \\
0.500 & 0.866 & 0.000 & 5.0 \\
0.000 & 0.000 & 1.000 & 0.0 \\
0 & 0 & 0 & 1
\end{bmatrix}
\]

We now apply the transform (notice how the B’s ”cancel”):

\[
\begin{align*}
\begin{bmatrix}
A \\
P
\end{bmatrix}
&= \begin{bmatrix}
A \\
T
\end{bmatrix}^B P \\
&= \begin{bmatrix}
A \\
T
\end{bmatrix}
\begin{bmatrix}
3 \\
7 \\
0 \\
1
\end{bmatrix} \\
&= \begin{bmatrix}
9.098 \\
12.562 \\
0 \\
1
\end{bmatrix}
\end{align*}
\]
We have used transforms to relate the mapping of a point from one frame to another. We can also view a transform as an operator which translates and/or rotates points. For example, let's say we have a point $A P_1 = [3 \ 7 \ 0]^T$ and we wish to rotate it by 30° about $\hat{Z}$ and translate it by $[10 \ 5 \ 0]^T$. Now we wish to find the transformed point $A P_2$, pictured below:
This example is identical to the previous one, except that we view this as an operator, not as a mapping. In this case, there is only one frame, $A$. The transformation matrix is the same as before, although we have no preceding super- or subscripts because the operation takes place within the same frame.

$$
T = \begin{bmatrix}
0.866 & -0.500 & 0.000 & 10.0 \\
0.500 & 0.866 & 0.000 & 5.0 \\
0.000 & 0.000 & 1.000 & 0.0 \\
0 & 0 & 0 & 1
\end{bmatrix}
$$

$$
A P_2 = T^A P_1
$$

$$
= T \begin{bmatrix}
3 \\
7 \\
0 \\
1
\end{bmatrix}
= \begin{bmatrix}
9.098 \\
12.562 \\
0 \\
1
\end{bmatrix}
$$
Considering the above with $^C P$ given, we can express this point in frames $B$ and then $A$,

\[
B \, P = B \, T^C \, P \\
A \, P = A \, B \, T^B \, P \\
= A \, B \, T^B \, B \, T^C \, P
\]

We can define a direct $C \rightarrow A$ transform: $A \, T = A \, T^B \, T^C \, T$. 
We can express the transform of $D$ with respect to $U$ in two different ways:

\[
\begin{align*}
U_D T &= U_A T_D^A T & \text{and} & \quad U_D T &= U_B T_C^B T_D^C T
\end{align*}
\]
The direction of the arrows indicates how existing transforms are defined. Let's say that all transforms but $B_C T$ are known and we wish to solve for it.

\[ B_C T = U_B T^{-1} U_A T_D T_C T_D T^{-1} \]
Rotation matrices must be orthonormal, meaning that their rows and columns are orthogonal unit vectors. Even further, rotation matrices must have a determinant of +1. Cayley’s formula for orthonormal matrices states that any such matrix can be rewritten as follows:

\[
R = (I - S)^{-1}(I + S)
\]

where

\[
S = \begin{bmatrix}
0 & -s_z & s_y \\
s_z & 0 & -s_x \\
-s_y & s_x & 0
\end{bmatrix}
\]

The important point is that a rotation matrix actually has only three free parameters. In fact any 3-D rotation can be represented by as little as three numbers. We will look at a couple of example representations…
We can describe the rotation of a frame $B$ with respect to a frame $A$ as a sequence of rotations about $\hat{X}_A$, $\hat{Y}_A$, and $\hat{Z}_A$ (other orders beside X-Y-Z are possible). We are rotating about $A$ which is fixed.

The rotation angles are $\gamma$, $\beta$, and $\alpha$. We can generate a rotation matrix by combining all three:

$$A_B^A R_{XYZ}(\gamma, \beta, \alpha) = R_Z(\alpha) R_Y(\beta) R_X(\gamma)$$

$$= \begin{bmatrix}
  c\alpha & -s\alpha & 0 \\
  s\alpha & c\alpha & 0 \\
  0 & 0 & 1 \\
\end{bmatrix} \begin{bmatrix}
  c\beta & 0 & s\beta \\
  0 & 1 & 0 \\
  -s\beta & 0 & c\beta \\
\end{bmatrix} \begin{bmatrix}
  1 & 0 & 0 \\
  0 & c\gamma & -s\gamma \\
  0 & s\gamma & c\gamma \\
\end{bmatrix}$$
Euler angles are defined by rotating the "moving" frame, $B$, about its own axes. The following is for the order Z-Y-X:

We can view this as a series of rotations: $A \rightarrow B'$, $B' \rightarrow B''$, and $B'' \rightarrow B$, where $B'$ and $B''$ are intermediary frames.

$$\begin{align*}
A_B R &= A_{B'} R_{B''}^{B'} R_{B''}^{B} R
\end{align*}$$

From this perspective the rotation by $\gamma$ happens first, then $\beta$, then $\alpha$. Interestingly, this is the same order and the same angles as for the X-Y-Z fixed angle representation—the two are equivalent!
So the X-Y-Z fixed angle convention is equivalent to the Z-Y-X Euler angle convention. There are 12 equivalent pairs of conventions.

We also have the axis-angle representation which represents the rotation via an equivalent axis of rotation specified by a unit vector $\hat{N}$ (2 free parameters) and an angle $\theta$.

Finally, the unit quaternion representation (see additional notes) is cleanly defined from the axis-angle representation:

$$q = \cos \frac{\theta}{2} + \sin \frac{\theta}{2} \hat{N}$$

Note the difference between the representation of a rotation and a rotation operator. The only rotation operators we have seen our rotation matrices and unit quaternions.