Spatial Transforms

COMP 4766/6912: Autonomous Robotics

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- We will look at transforms in two different ways: as mappings and as operators
- We will discuss two different orientation representations: rotation matrices and quaternions

Notation

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The A superscript appears to the left of the actual point and means that we are describing the point with respect to A. The same point described in frame B would be ^{B}P.

Frame A is described by three mutually orthogonal unit vectors $\hat{X}_A$, $\hat{Y}_A$, and $\hat{Z}_A$. The components of ${}^{A}P$ are the projections of the vector corresponding to this point onto these axes.

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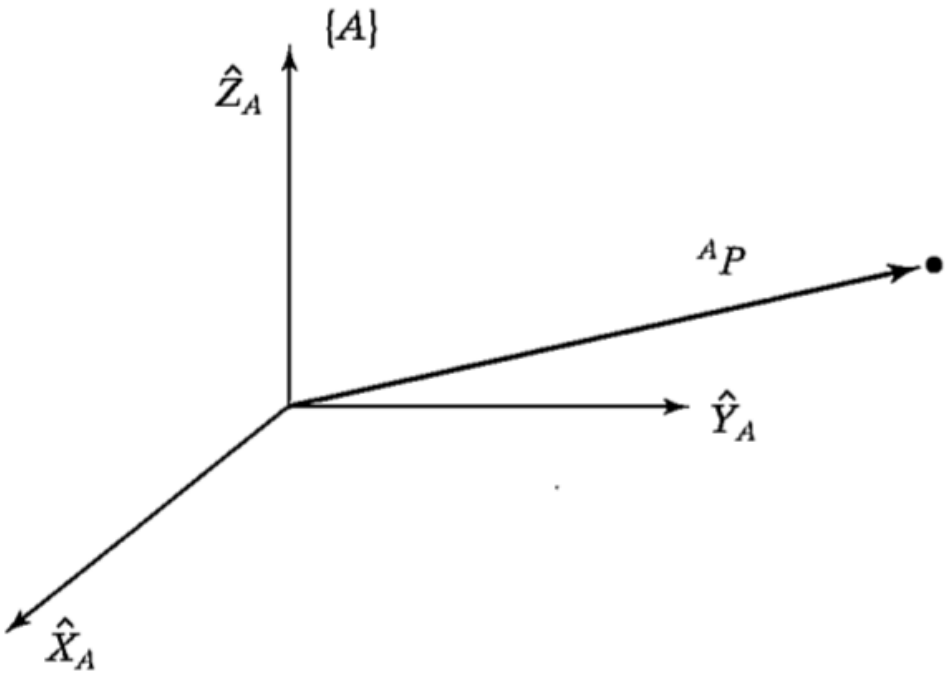

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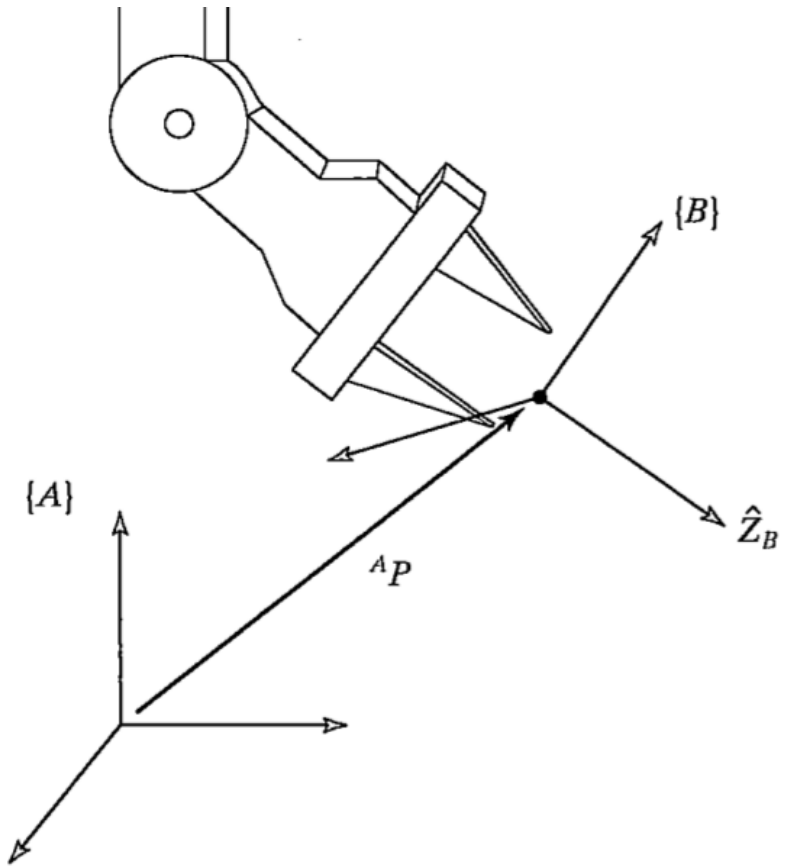

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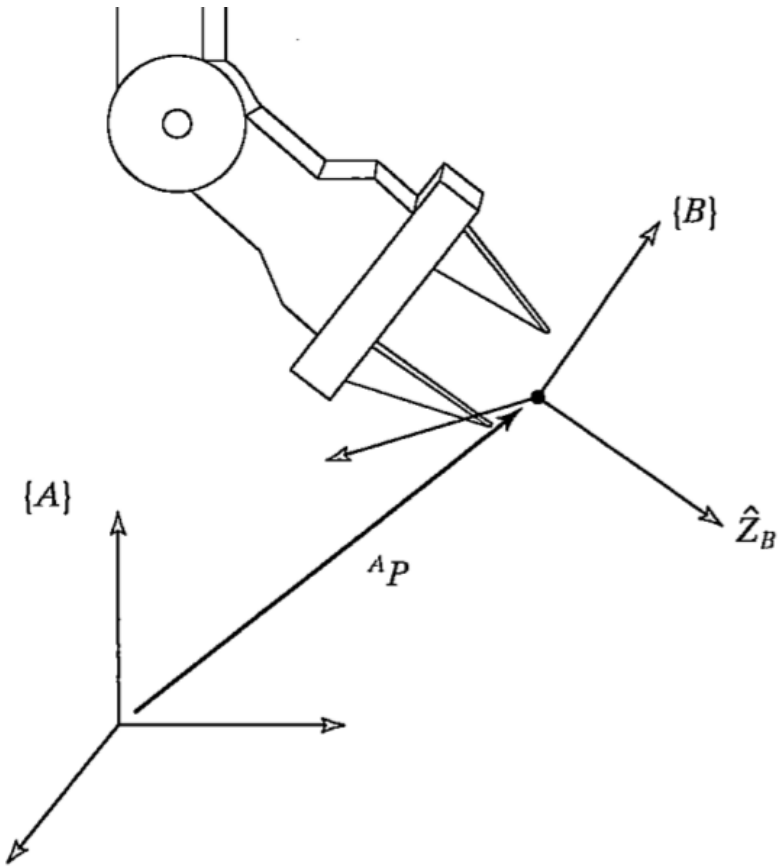

We can describe one frame w.r.t. to another by specifying the **rotation** and **translation** of the movement that separates them.

Rotation

The **rotation** of frame B w.r.t. frame A is defined through the mutually orthogonal unit vectors ${}^{A}\hat{X}_B$, ${}^{A}\hat{Y}_B$, and ${}^{A}\hat{Z}_B$. If we stack these three vectors as columns of a 3×3 matrix we get the rotation matrix that encodes the orientation of frame B w.r.t. frame A:

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$$ {}^{A}_{B}R = \begin{bmatrix} {}^{A}\hat{X}_B & {}^{A}\hat{Y}_B & {}^{A}\hat{Z}_B \end{bmatrix} = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix} $$

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$$ {}^A_B R = {}^B_A R^{-1} = {}^B_A R^T $$

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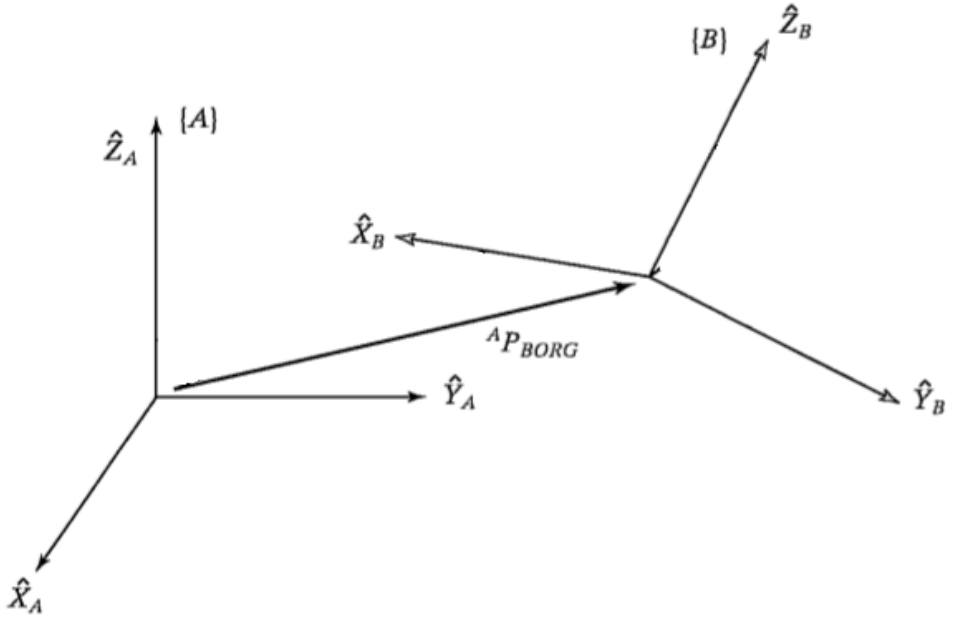

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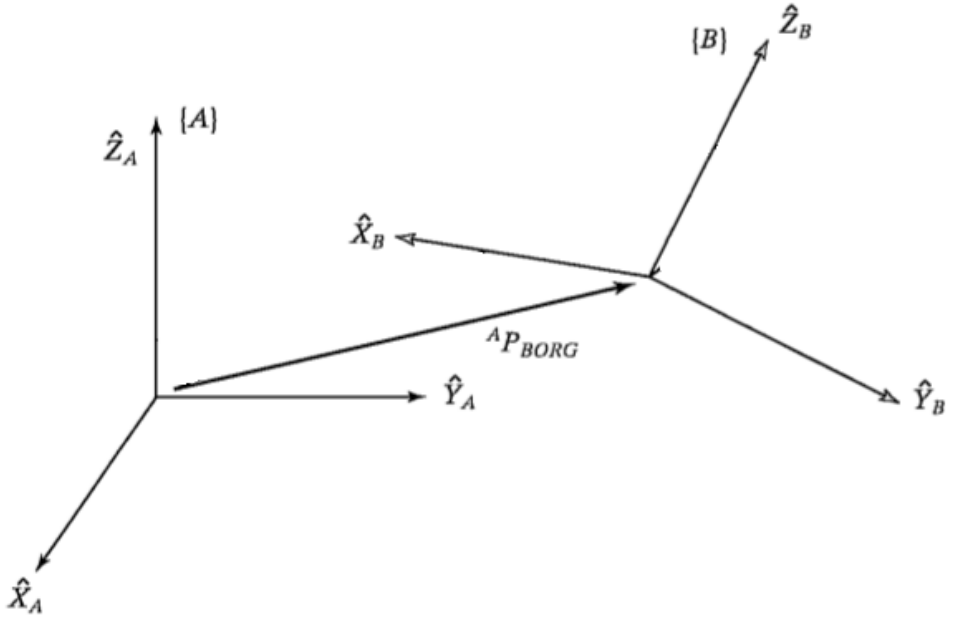

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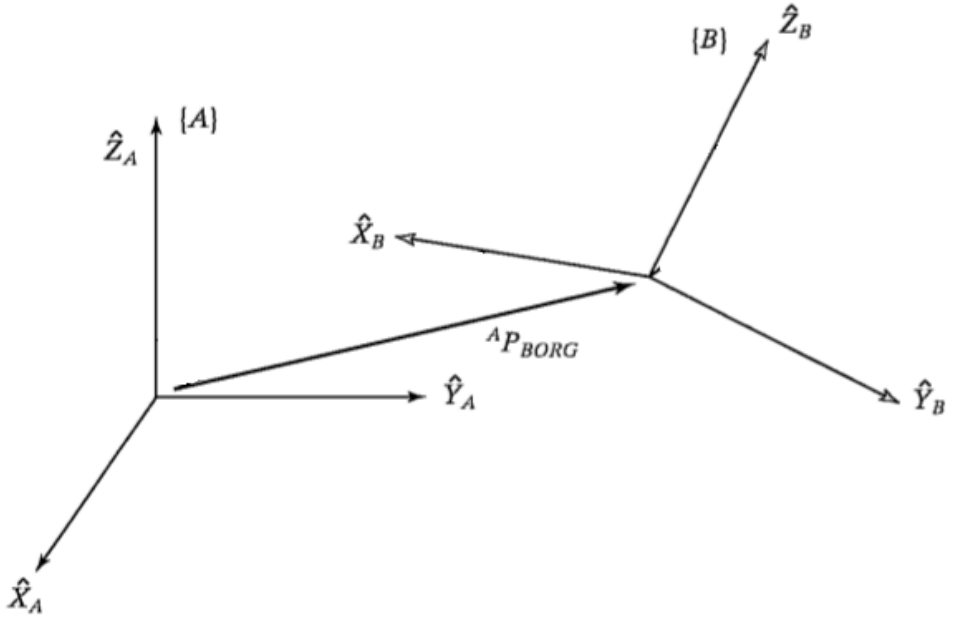

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$$\{B\} = \left\{ {}^{A}_{B}R, {}^{A}P_{BORG} \right\}$$

There is usually a universal (a.k.a. global or inertial) frame that others are defined w.r.t.. However, its often the case that we don't have a direct description of a frame with respect to the universal (e.g. for frame C below).

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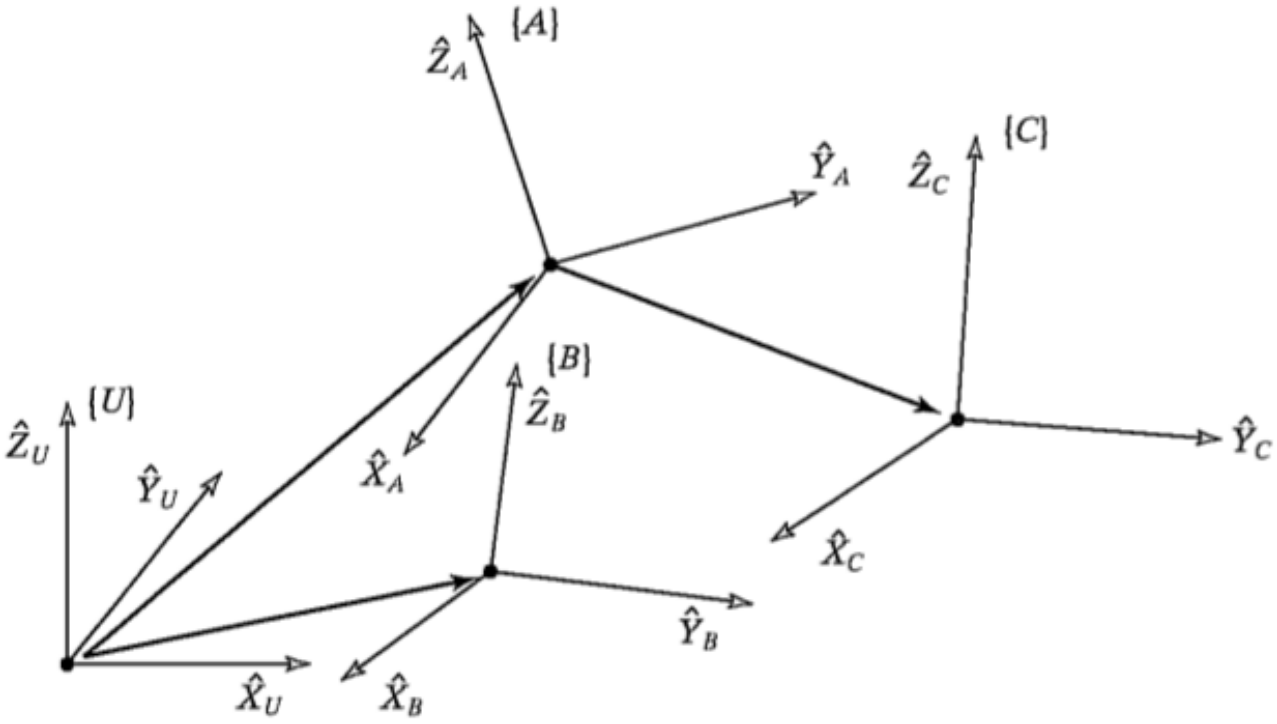

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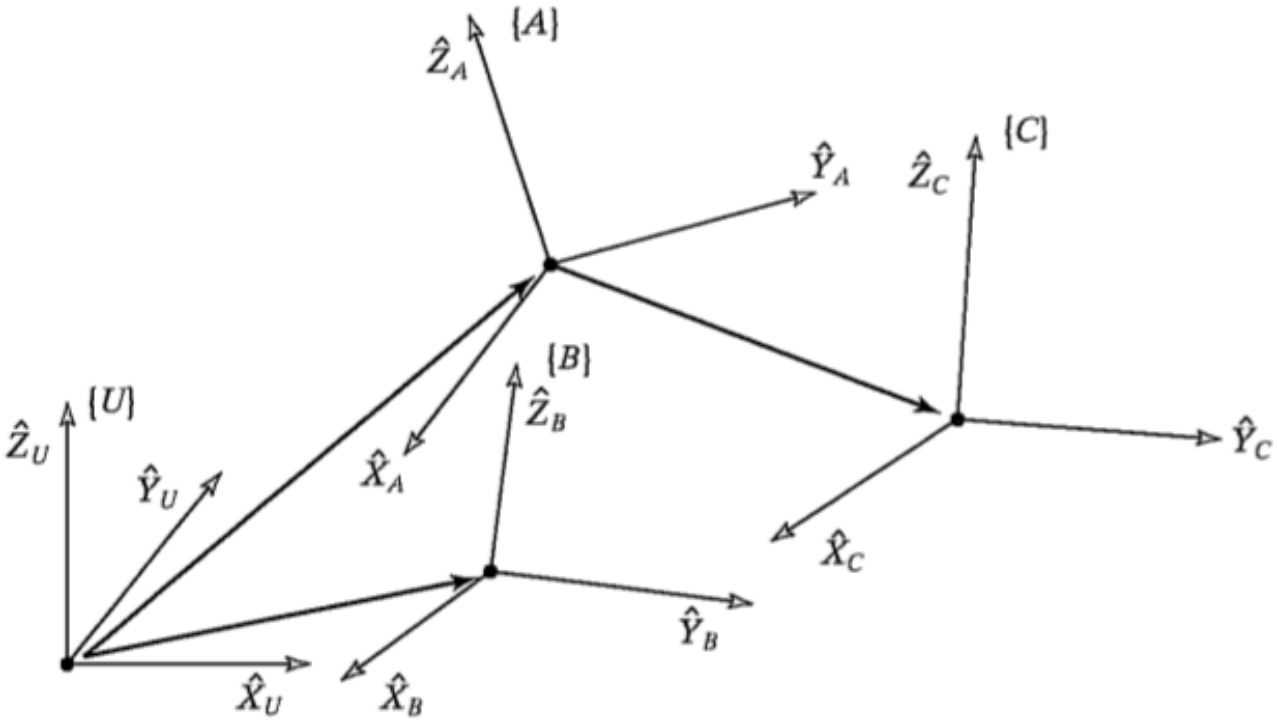

We would like a way of mapping descriptions of points from one frame to another.

Mappings

A mapping is a change in our description of the same entity from one frame to another. Lets say we know the position of a point w.r.t. B but we want to compute it w.r.t. A. For example, we wish to find ${}^{A}P$ below:

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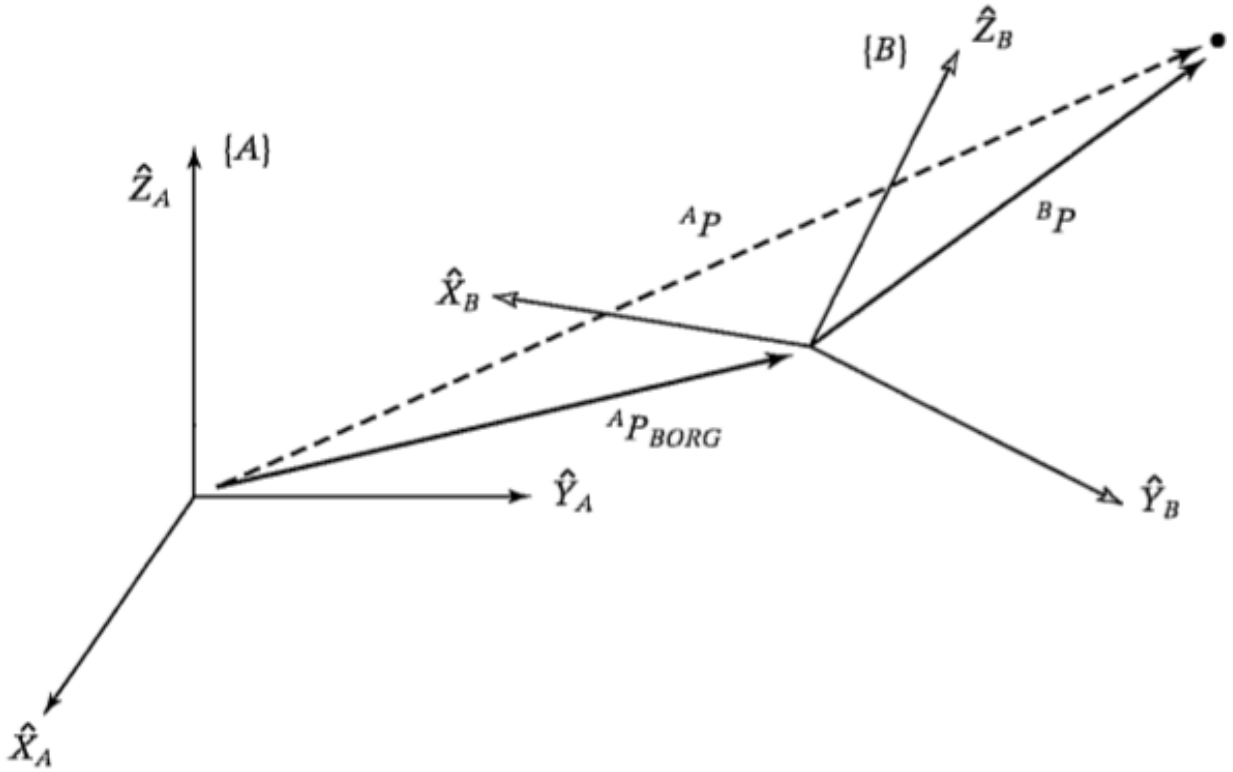
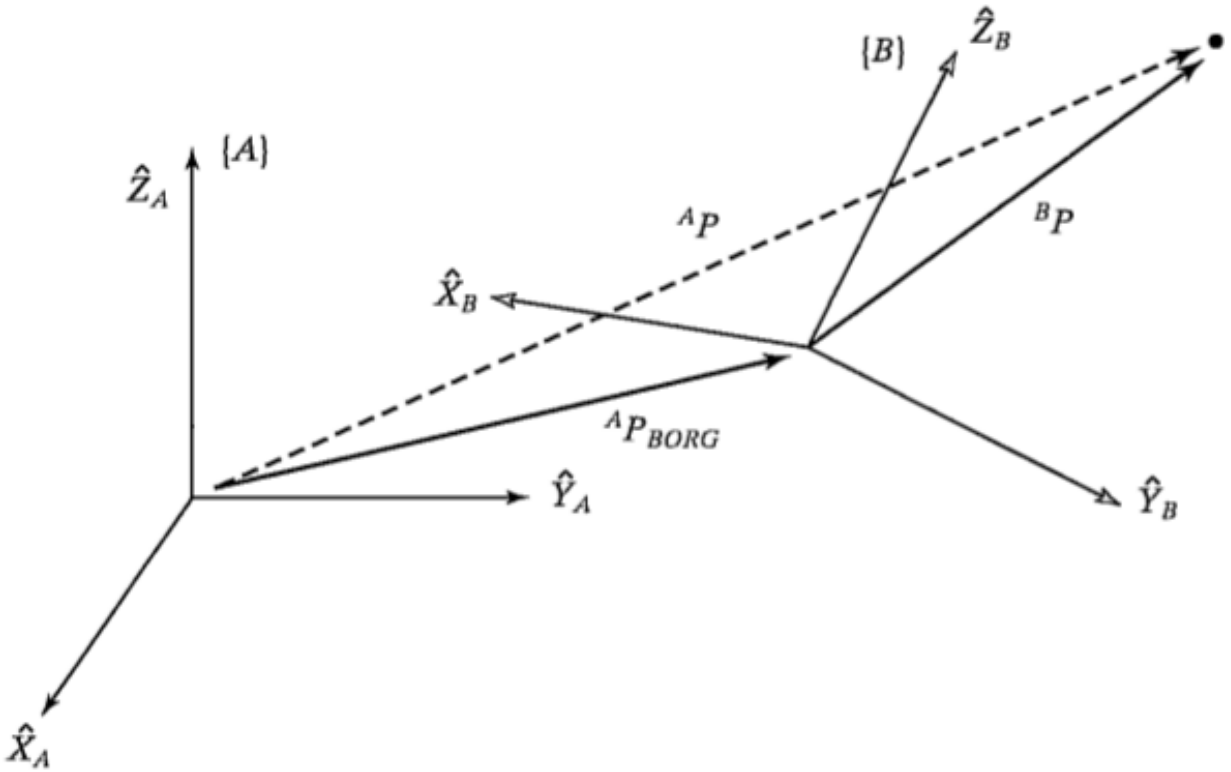
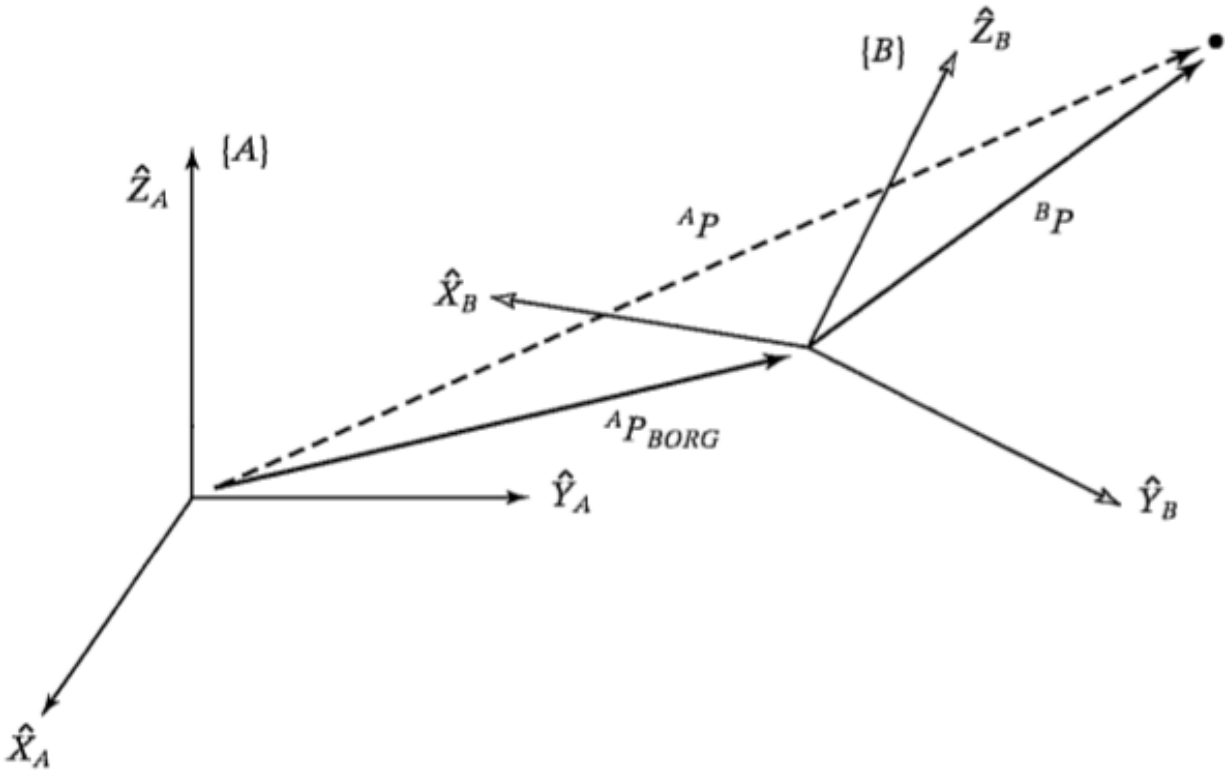




If we have ${}^{B}P$ and our description of B we can get ${}^{A}P$.



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$$\begin{bmatrix} {}^{A}P \\ 1 \end{bmatrix} = \left[\begin{array}{ccc|c} & {}^{A}_{B}R & & {}^{A}P_{BORG} \\ \hline 0 & 0 & 0 & 1 \end{array}\right] \begin{bmatrix} {}^{B}P \\ 1 \end{bmatrix}$$

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This yields the equation at the top of this slide, as well as $1 = 1$ (uninformative, but reassuring). We can represent other sorts of transformations (e.g. shearing, scaling, perspective projection) by modifying particular entries in the augmented matrix.

2-D Example

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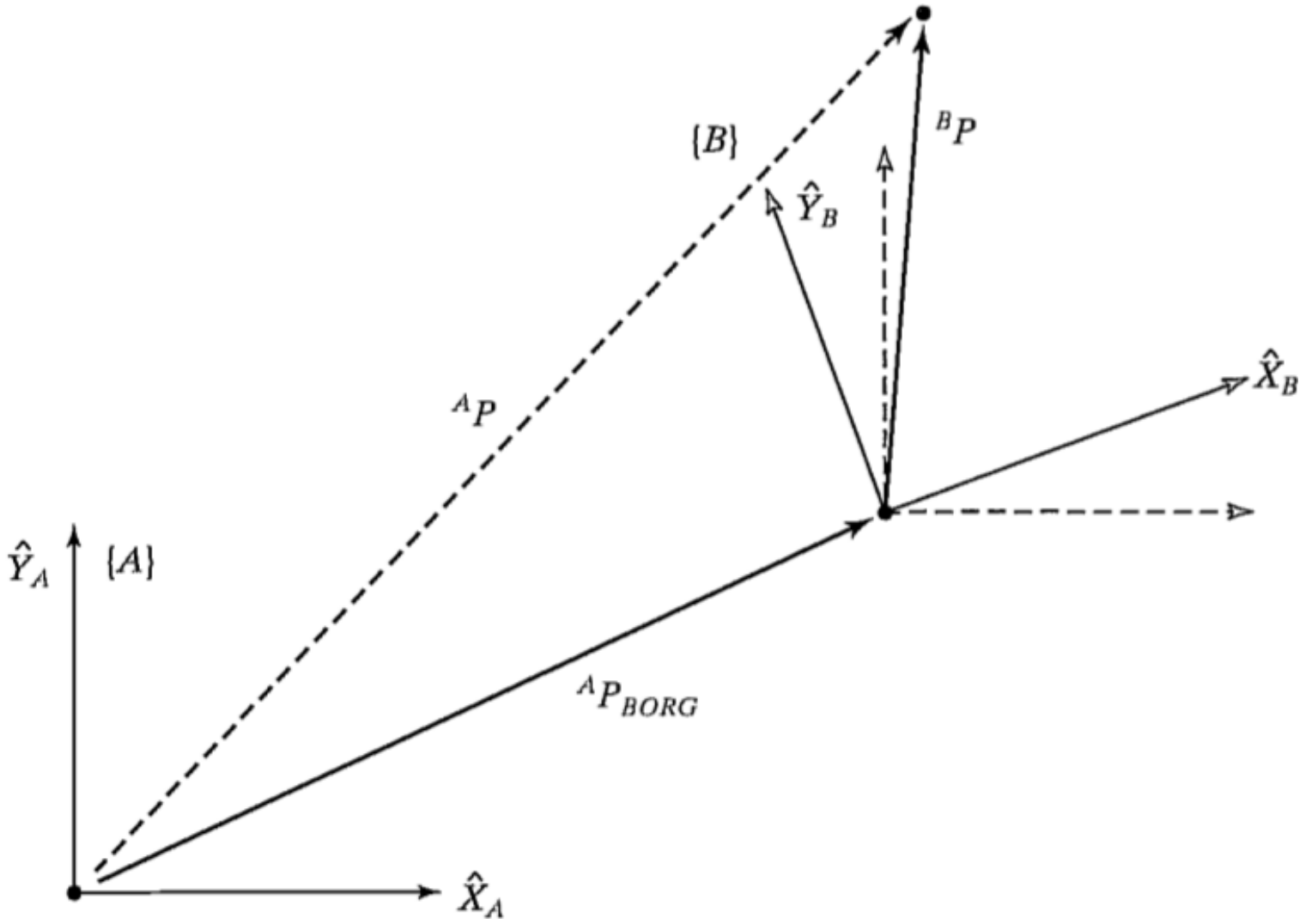

We get the following transform matrix:

$$ {}^{A}_{B}T = \begin{bmatrix} 0.866 & -0.500 & 0.000 & 10.0 \\ 0.500 & 0.866 & 0.000 & 5.0 \\ 0.000 & 0.000 & 1.000 & 0.0 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

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Transform as Operator

We have used transforms to relate the mapping of a point from one frame to another. We can also view a transform as an **operator** which translates and/or rotates points. For example, lets say we have a point ${}^{A}P_1 = [3\ 7\ 0]^T$ and we wish to rotate it by 30° about $\hat{Z}$ and translate it by $[10\ 5\ 0]^T$. Now we wish to find the transformed point ${}^{A}P_2$, pictured below:

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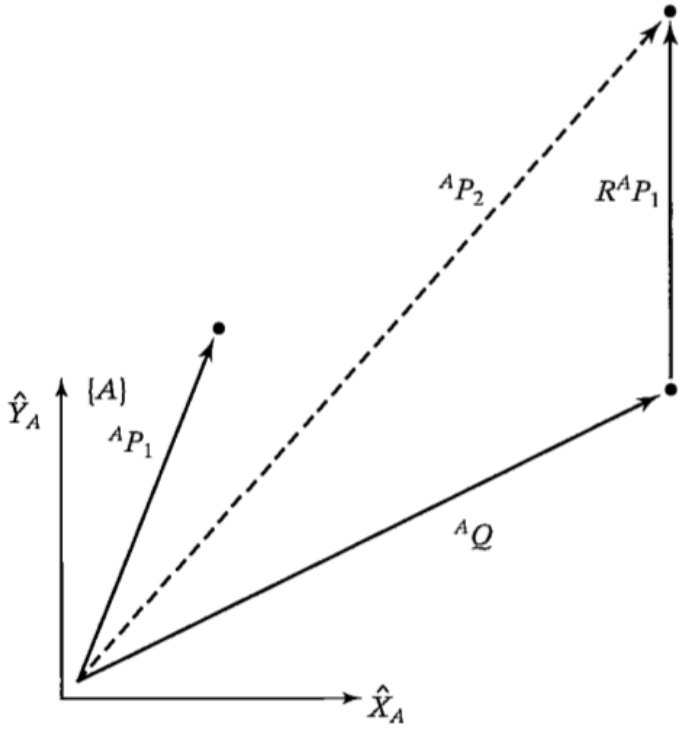

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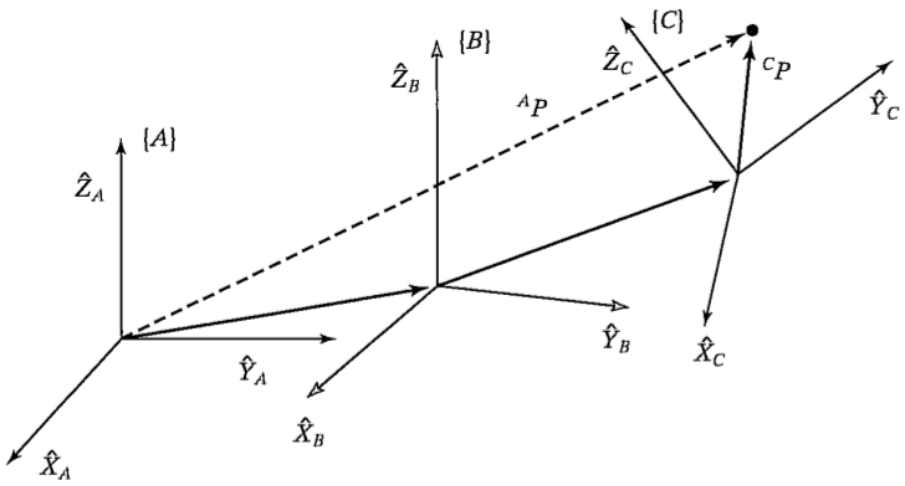

Considering the above with ${}^{C}P$ given, we can express this point in frames B and then A,

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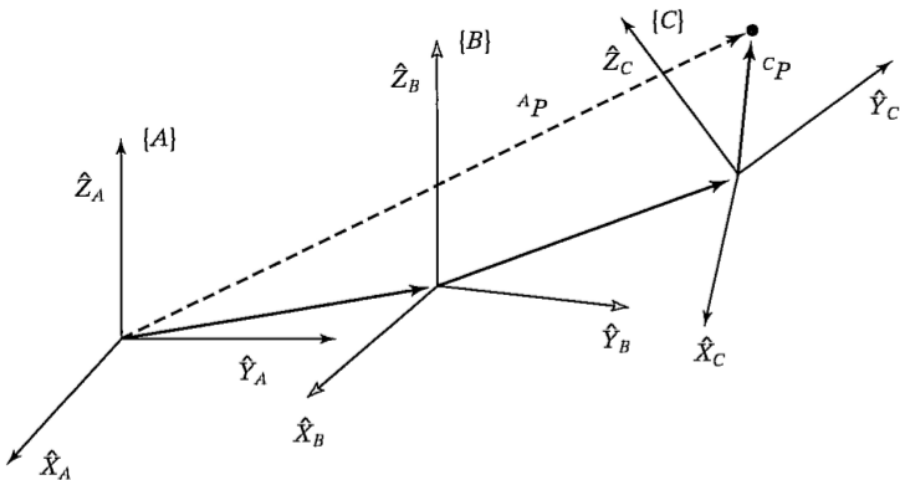

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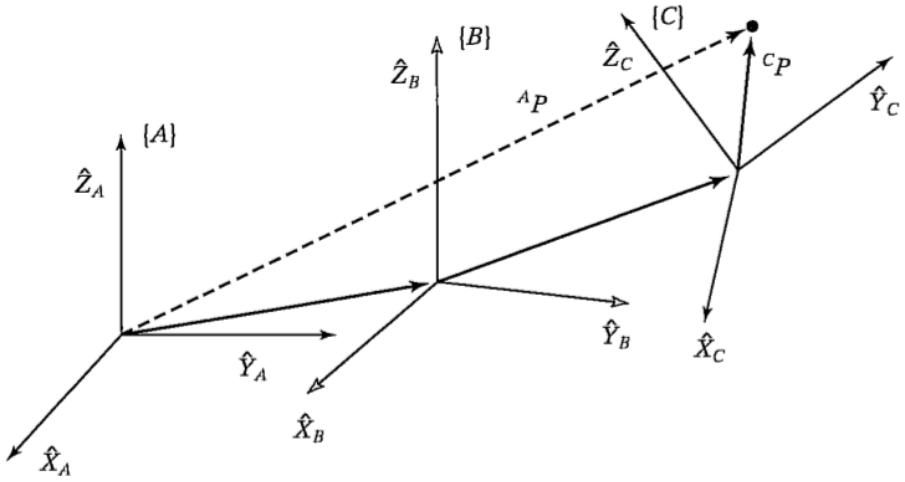

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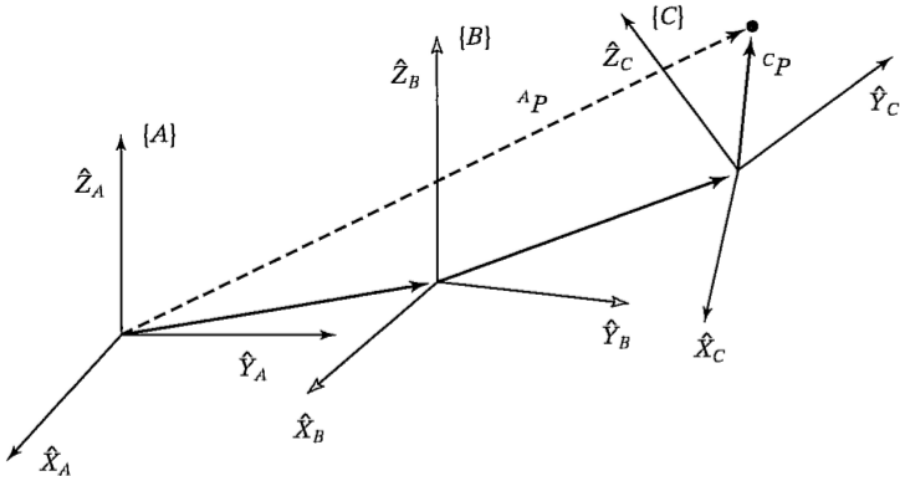

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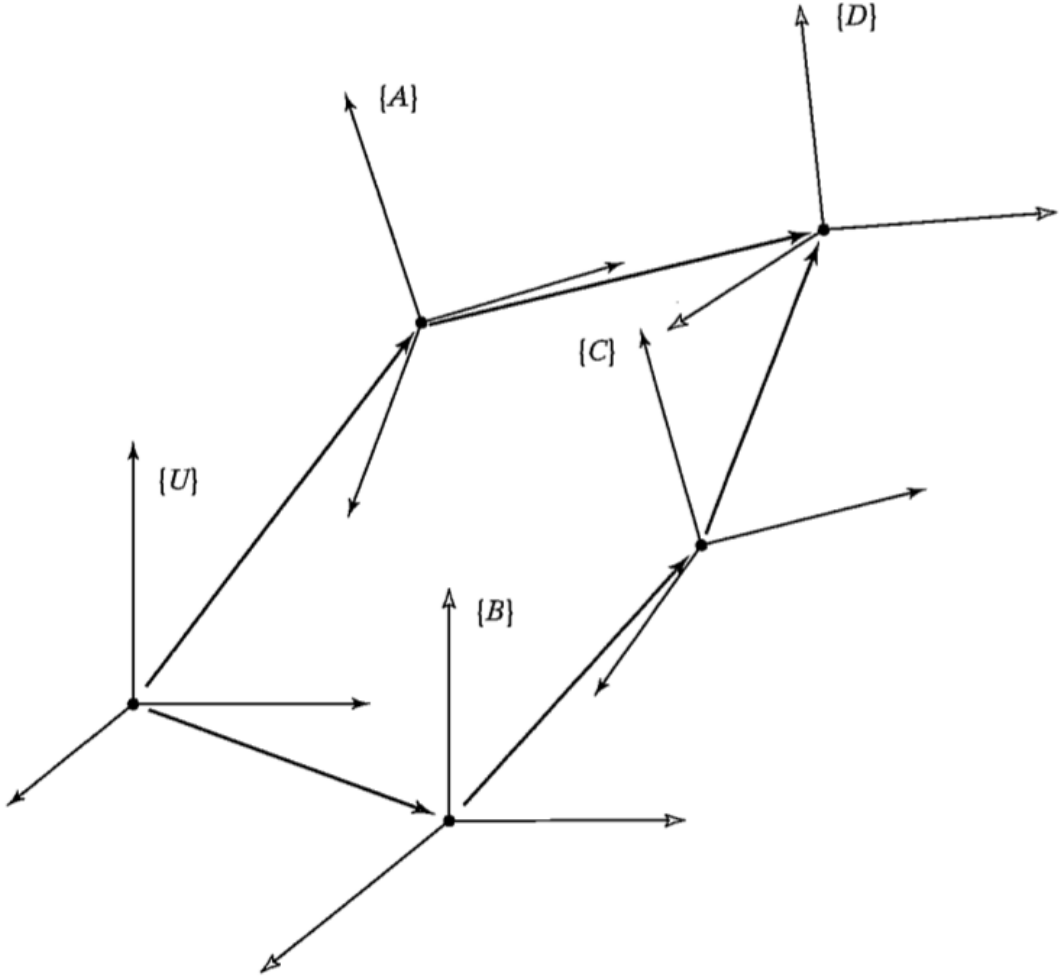
We can define a direct $C \rightarrow A$ transform: ${}^{A}_{C}T = \left({}^{A}_{B}T\right)\left({}^{B}_{C}T\right)$.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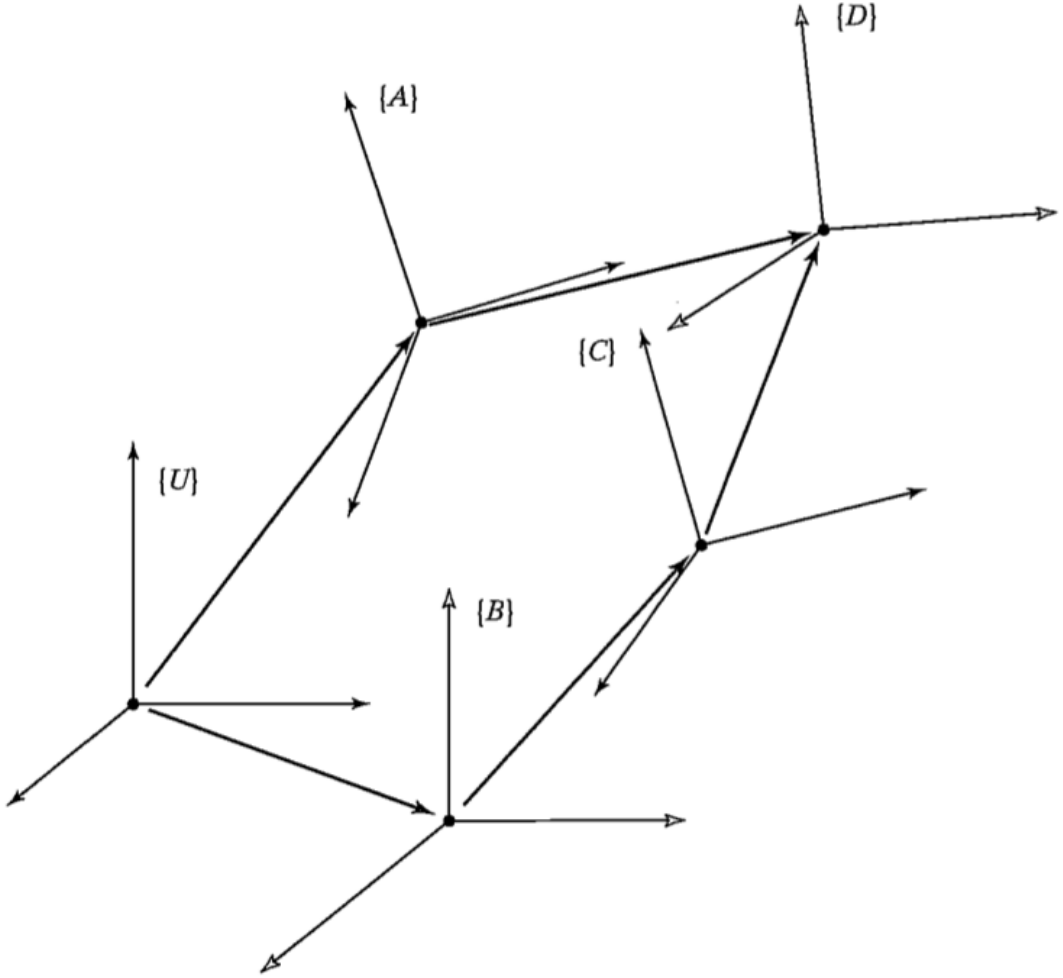
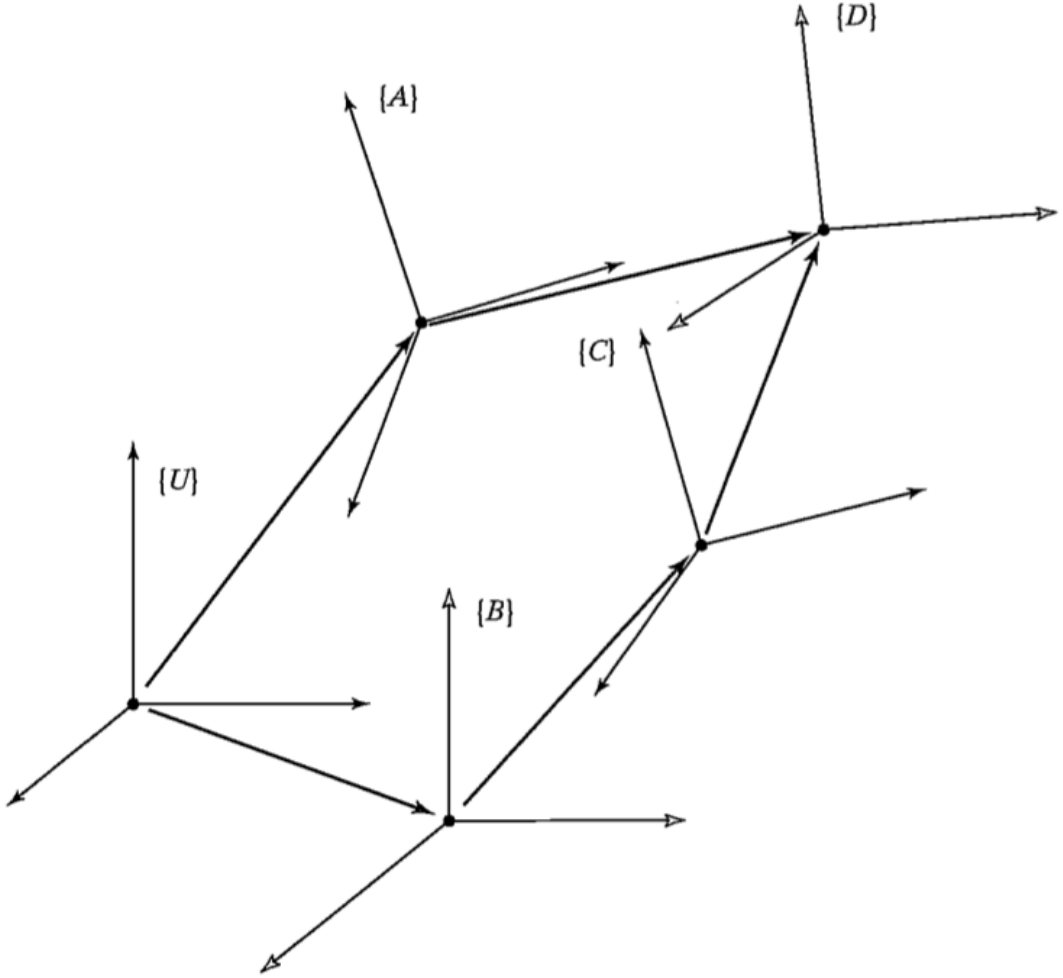


We can express the transform of D with respect to U in two different ways:

$$ {}^{U}_{D}T = \left({}^{U}_{A}T\right)\left({}^{A}_{D}T\right) $$



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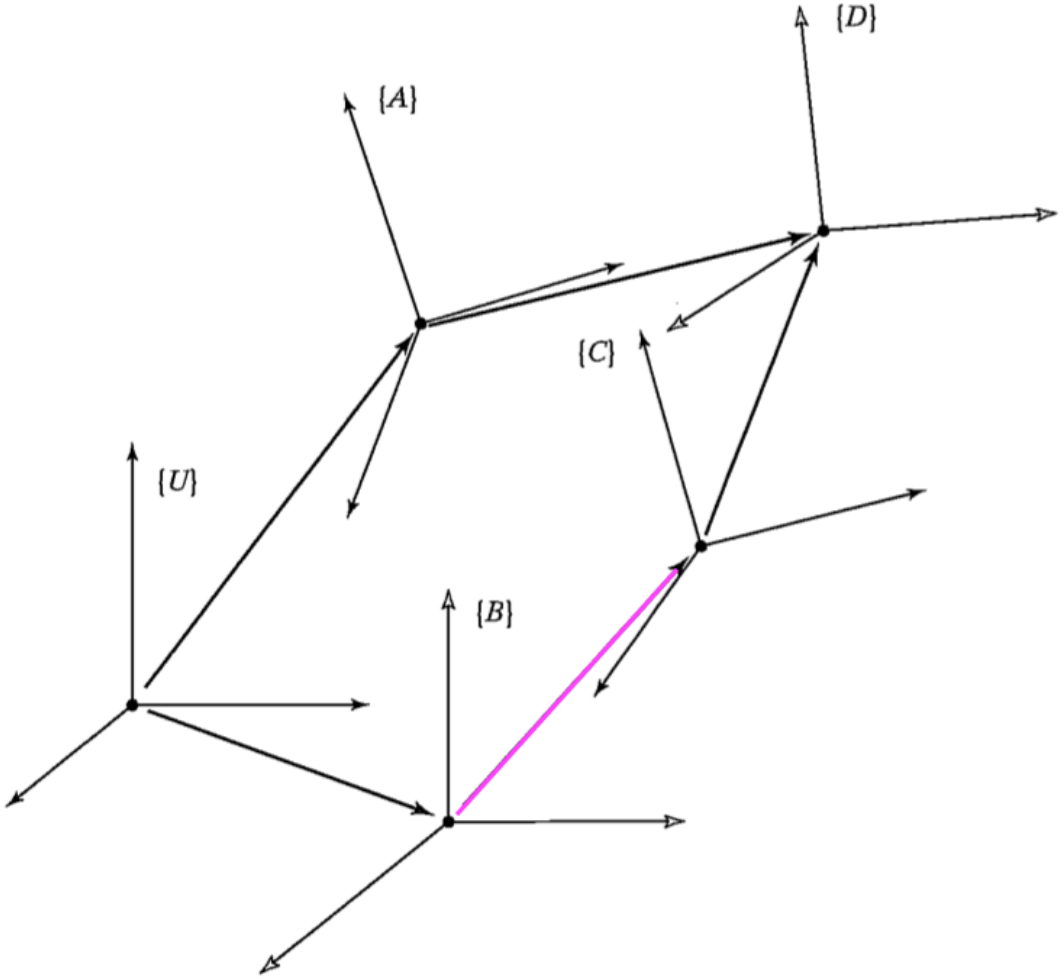

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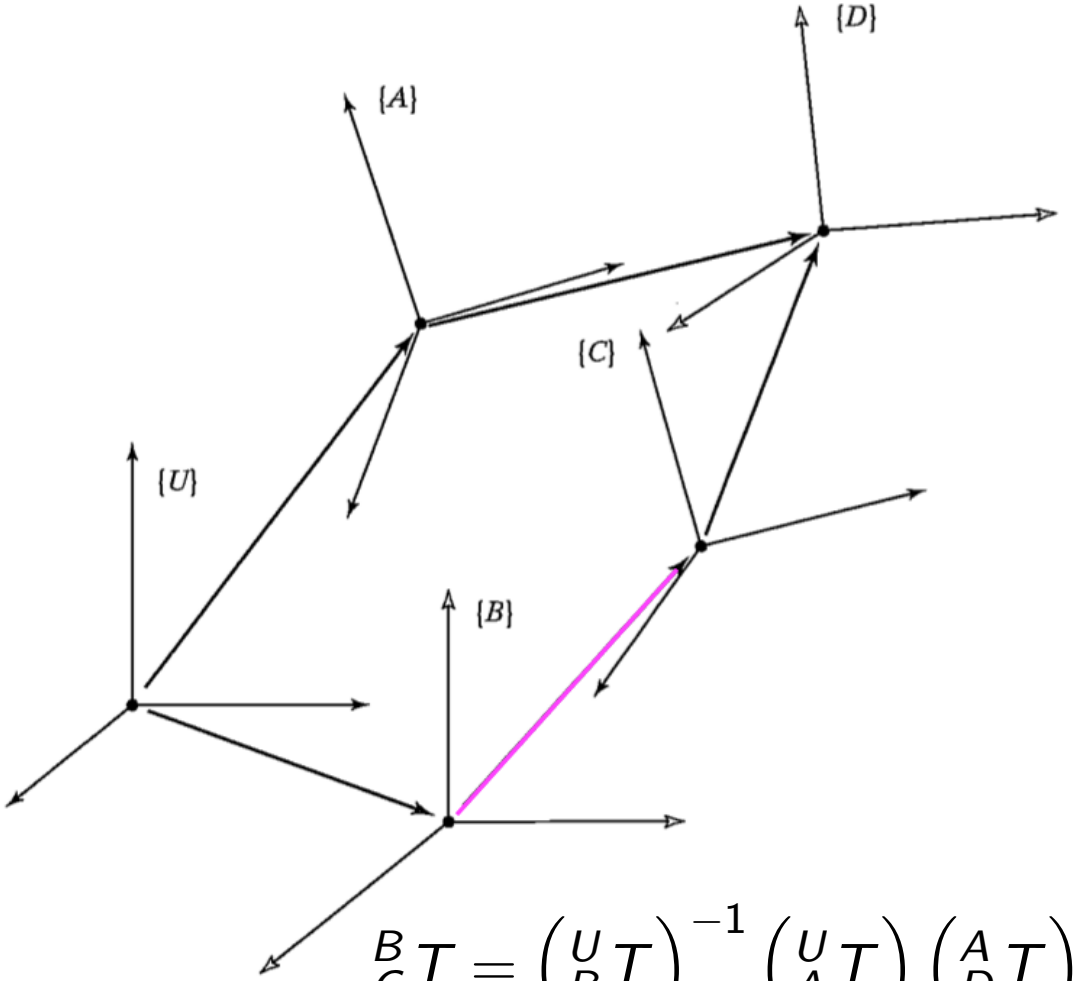

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The important point is that a rotation matrix actually has only three free parameters. In fact any 3-D rotation can be represented by as little as three numbers. We will look at a couple of example representations...

X-Y-Z Fixed Angles

We can describe the rotation of a frame B with respect to a frame A as a sequence of rotations about $\hat{X}_A$, $\hat{Y}_A$, and $\hat{Z}_A$ (other orders beside X-Y-Z are possible). We are rotating about A which is fixed.

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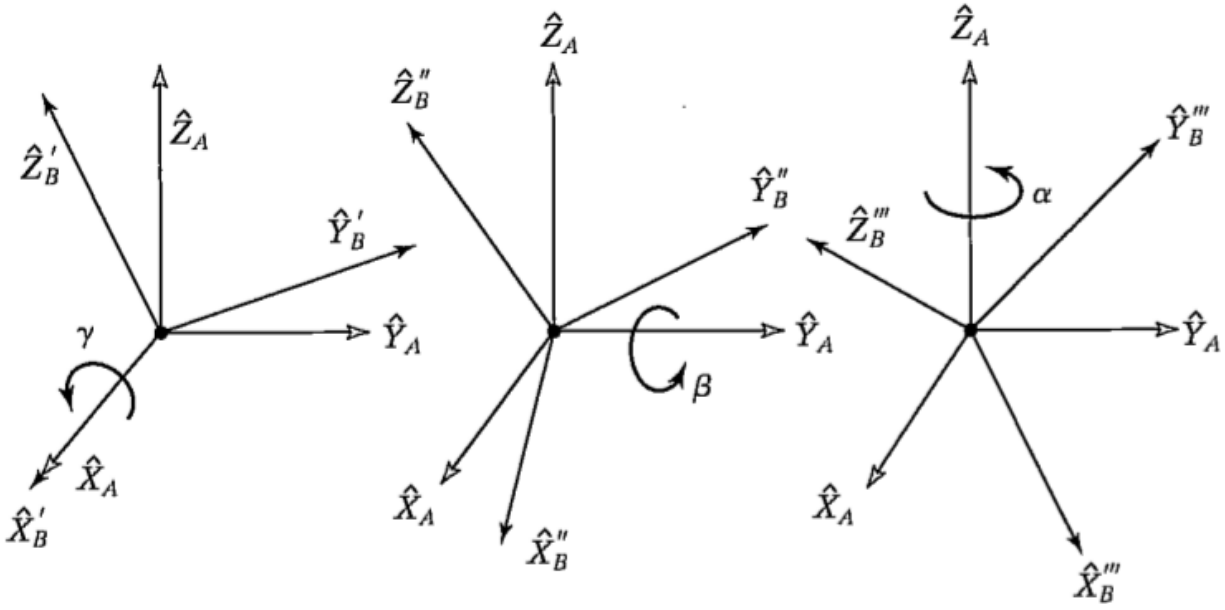

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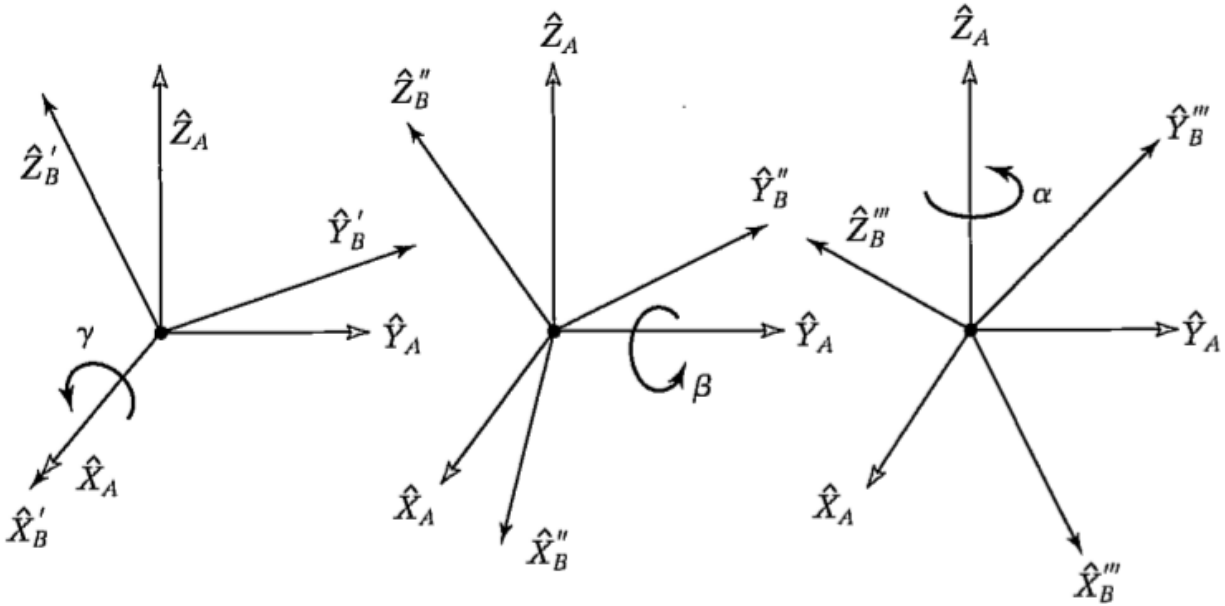

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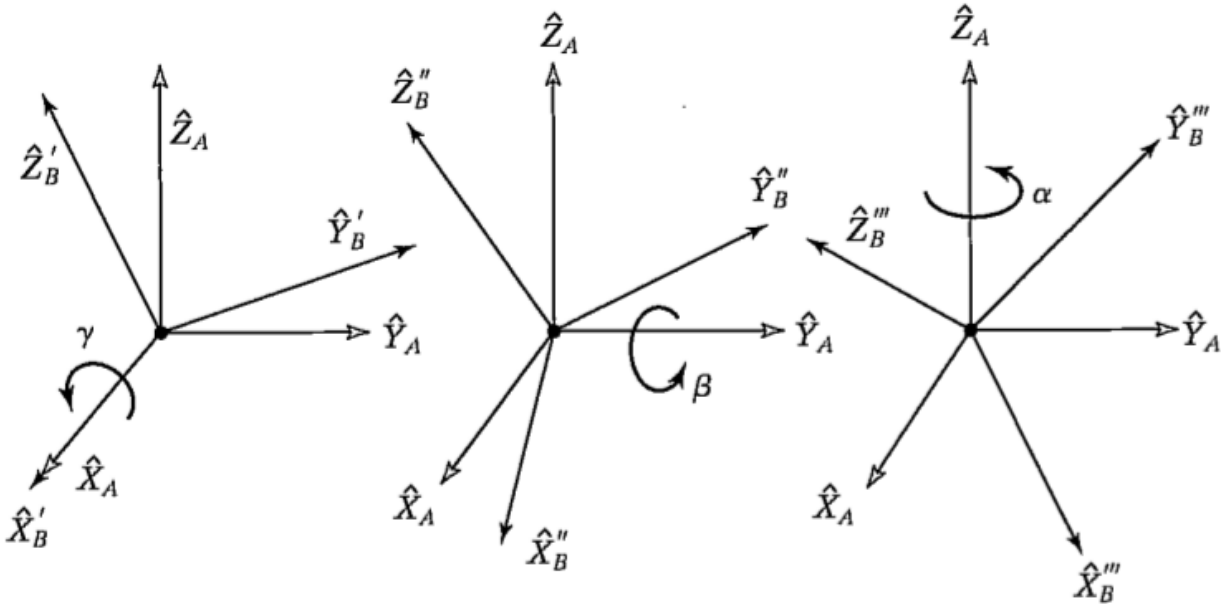

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$$
\begin{aligned}
{}^A_B R_{XYZ}(\gamma, \beta, \alpha) &= R_Z(\alpha) R_Y(\beta) R_X(\gamma) \\
&= \begin{bmatrix} c\alpha & -s\alpha & 0 \\ s\alpha & c\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c\beta & 0 & s\beta \\ 0 & 1 & 0 \\ -s\beta & 0 & c\beta \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & c\gamma & -s\gamma \\ 0 & s\gamma & c\gamma \end{bmatrix}
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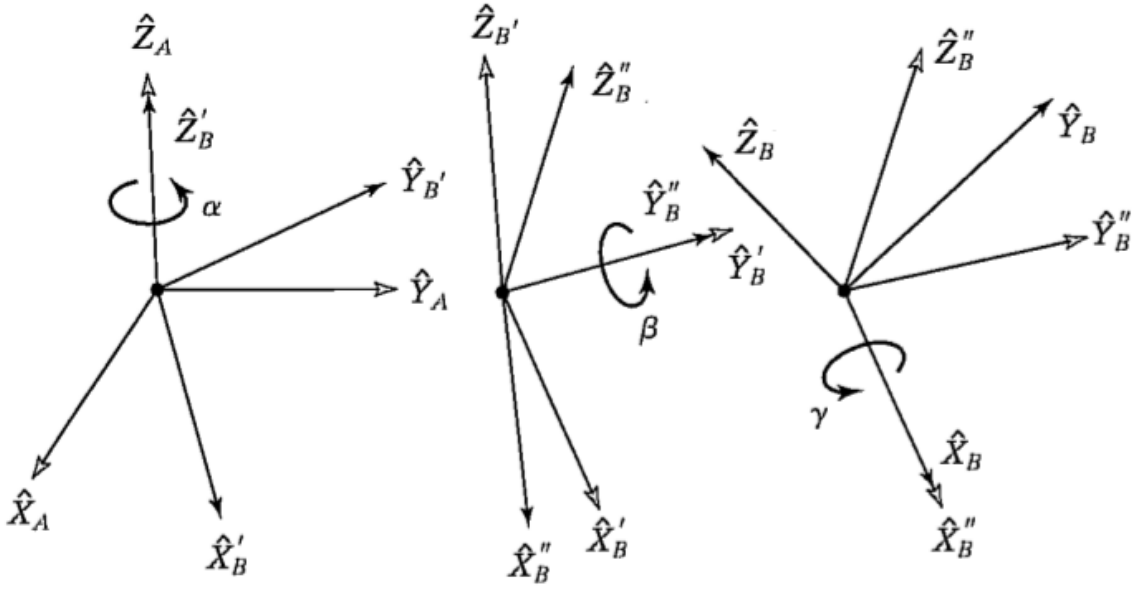

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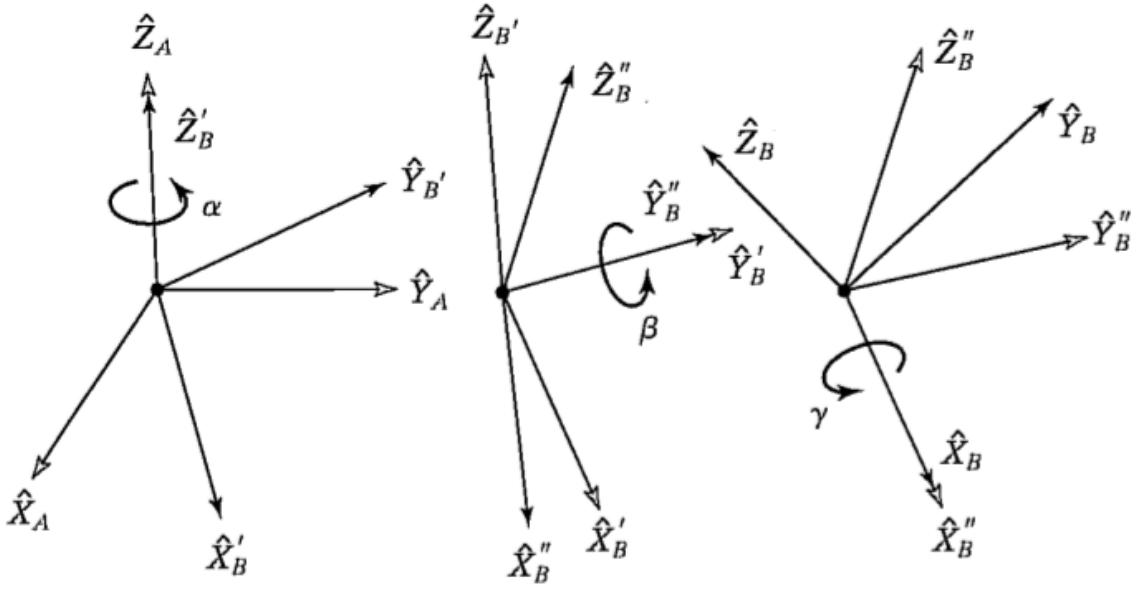

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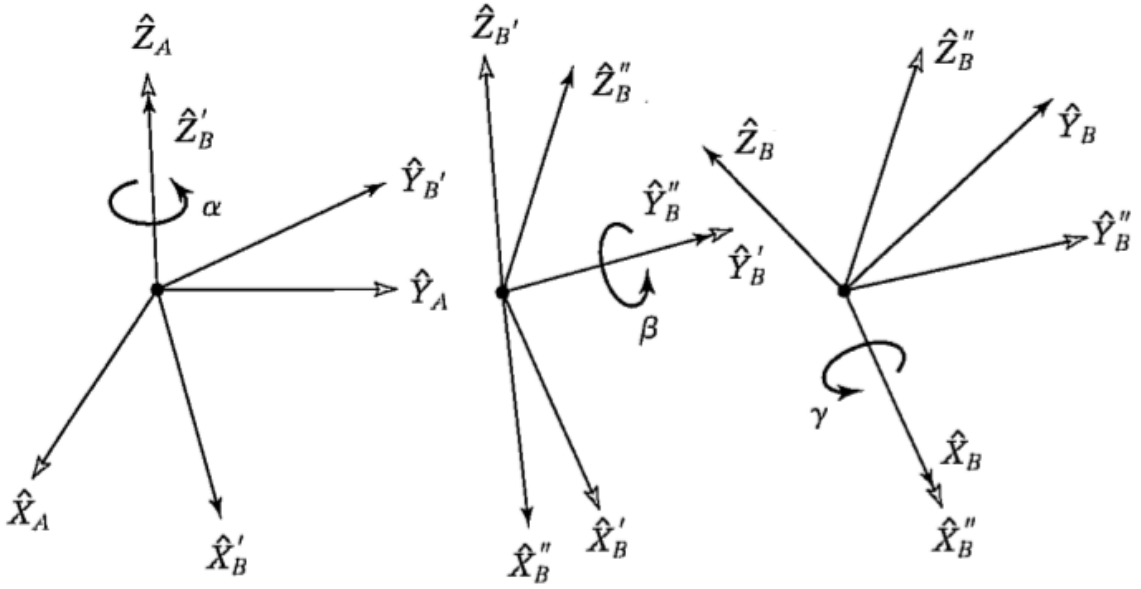

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From this perspective the rotation by γ happens first, then β, then α. Interestingly, this is the same order and the same angles as for the X-Y-Z fixed angle representation—the two are equivalent!

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Unit Quaternions and the Axis-Angle Representation

- SEE SEPARATE NOTES ON QUATERNIONS
- Note that the relationship between rotation and unit quaternions comes through the **axis-angle** representation of rotation:
  - The rotation is represented via a single axis of rotation specified by a unit vector $\hat{N}$ (2 free parameters) and an angle θ.
- Unit quaternions are related to these quantities as follows:

$$q = \cos\frac{\theta}{2} + \sin\frac{\theta}{2}\hat{N}$$