Planning: Part 2
Probabilistic Planning

Computer Science 6912

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![Diagram of a near-symmetric environment with narrow and wide corridors. The robot starts at the center with unknown orientation. Its task is to move to the goal location on the left.](image)

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Since the robot’s actions are uncertain, it is helpful to have a complete plan that covers the whole state space in case the robot wanders off the ideal route. A control policy gives the right action to perform in any state. A control policy is also known as a universal plan.
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Notice that the longer path is now preferred since it reduces the risk of running into a wall.
Value Iteration

We require an algorithm to evaluate the value of all possible states so that we can choose the best next state from any current state (i.e. the best movement).

For example, we can define a payoff function $r(x, u) = \begin{cases} +100 & \text{if we expect } u \text{ to bring us to the goal state} \\ -1 & \text{otherwise} \end{cases}$

We get a reward of 100 for reaching the goal state and a movement penalty of $-1$. We attempt to maximize the expected future payoff, which will hopefully lead us to the goal state while minimizing future penalties.
We require an algorithm to evaluate the value of all possible states so that we can choose the best next state from any current state (i.e. the best movement). The algorithm used here is known as **value iteration**.

We first need to define a *payoff function* to get the immediate expected reward or penalty for any movement \( u \) in state \( x \).

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How do we measure the value of future payoffs?

Define a planning horizon of $T$ time steps and maximize payoffs within this time frame.

1. $T = 1$: The robot is very short-sighted and attempts only to maximize the immediate payoff. This is known as the greedy case.

2. $T > 1$: The robot attempts to maximize payoff over the finite planning horizon $T$. However, determining the best value for $T$ may be difficult.

3. $T = \infty$: Clearly we cannot make plans infinitely far in advance. Instead we define a discount factor $\gamma < 1$ which is used to discount future payoffs. Future payoffs are discounted by $\gamma^\tau$ where $\tau$ goes from $1 \to \infty$. In practice we only plan ahead up to the point where $\gamma^\tau$ becomes negligible.

The value iteration algorithm uses $T = \infty$ with future payoffs discounted.
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Our fundamental goal is to define a control policy which is a mapping from states to actions:

$$\pi(x) \rightarrow u$$

We wish to determine the policy $\pi$ that maximizes future cumulative payoff.

Consider first the optimal policy for $T = 1$:

$$\pi_1(x) = \arg\max_u r(x,u)$$

($\arg\max_a f(a)$ is the value for $a$ at which $f(a)$ attains its maximum value. It could be a set if there are ties for the maximum.)

We define a value function $V_T(x)$ which gives the expected payoff for the current policy, discounted by $\gamma$.

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\[
\pi_2(x') = \arg\max_u \left[ r(x,u) + \int V_1(x') p(x'|u,x) \, dx' \right]
\]

Remember that $r(x,u)$ gives the expected payoff for $u$, but the results of our actions are uncertain, which is why we must incorporate the expected value for all possible next states $x'$. The value function for $T = 2$ is,

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We can infer the general form for the optimal policy and associated value function for any $T$, 

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Since $V_T(x)$ is defined recursively, you can see that the discount factor will be applied $T$ times. $\gamma$ is usually set to a value just less than one (e.g. 0.99). So $\gamma^T$ will become vanishingly small as $T$ increases. Thus, we expect $V_T(x)$ to converge to some finite value as $T \to \infty$. Hence, 

\[
V_\infty(x) = \gamma \max_u \left[ r(x, u) + \int V_\infty(x') p(x'|u, x) dx' \right]
\]

This expression leads to a concrete algorithm which progressively refines the value function for increasing values of $T$. Assume that the value function is stored in a discrete grid. $V(x_i)$ represents the value function for discrete state $x_i$. 
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Hence, $V_\infty(x) = \gamma \max_u \left[ r(x, u) + \int V_{\infty}(x') p(x'|u, x) dx' \right]$.

This expression leads to a concrete algorithm which progressively refines the value function for increasing values of $T$. Assume that the value function is stored in a discrete grid. $V(x_i)$ represents the value function for discrete state $x_i$. 
We can infer the general form for the optimal policy and associated value function for any $T$,

$$
\pi_T(x) = \arg\max_u \left[ r(x, u) + \int V_{T-1}(x') p(x' | u, x) dx' \right] \\
V_T(x) = \gamma \max_u \left[ r(x, u) + \int V_{T-1}(x') p(x' | u, x) dx' \right]
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Algorithm

discreteValueIteration()

for i = 1 to N
    \( V(x_i) = r_{min} \)
end for

repeat until convergence
    for i = 1 to N
        \( V(x_i) = \gamma \max_u \left[ r(x_i, u) + \sum_{j=1}^{N} V(x_j)p(x_j|u, x_i) \right] \)
    end for
end repeat

controlPolicy(x_i, V)
return \( \arg\max_u \left[ r(x_i, u) + \sum_{j=1}^{N} V(x_j)p(x_j|u, x_i) \right] \)
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    end for
end repeat

controlPolicy(x_i, V)

return \( \arg\max_u \left[ r(x_i, u) + \sum_{j=1}^{N} V(x_j)p(x_j|u, x_i) \right] \)
Figure 14.4  An example of an infinite-horizon value function $T_\infty$, assuming that the goal state is an “absorbing state.” This value function induces the policy shown in Figure 14.2a.
Figure 14.5 Example of value iteration over state spaces in robot motion. Obstacles are shown in black. The value function is indicated by the gray shaded area. Greedy action selection with respect to the value function lead to optimal control, assuming that the robot’s pose is observable. Also shown in the diagrams are example paths obtained by following the greedy policy.
Figure 14.6  (a) 2-DOF robot arm in an environment with obstacles. (b) The configuration space of this arm: the horizontal axis corresponds to the shoulder joint, and the vertical axis to its elbow joint configuration. Obstacles are shown in gray. The small dot in this diagram corresponds to the configuration on the left.
Figure 14.7  (a) Value iteration applied to a coarse discretization of the configuration space. (b) Path in workspace coordinates. The robot indeed avoids the vertical obstacle.