Lecture 7.
Quaternions

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Mechanics of Manipulation
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Today's outline

Overview, motivation

Background

Definition and properties

Rotation using unit quaternions

~~Intuition~~

~~Using quaternions to represent rotations~~

~~Why we love quaternions.~~

Motivation

Motivation

- Quaternions have nice geometrical interpretation.
- Quaternions have advantages in representing rotation.
- Quaternions are cool. Even if you don't want to use them, you might need to defend yourself from quaternion fanatics.

Why can't we invert vectors in $\mathbf{R}^3$?

- We can invert reals. $x \times \frac{1}{x} = 1$.
- We can invert elements of $\mathbf{R}^2$ using complex numbers. $z \times z^*/|z|^2 = 1$, where * is complex conjugate.
- Can we invert $\mathbf{v} \in \mathbf{R}^3$?
  - No.
- How about $\mathbf{v} \in \mathbf{R}^4$?
  - Yes! Hamilton's quaternions are to $\mathbf{R}^4$ what complex numbers are to $\mathbf{R}^2$.

Complex numbers versus quaternions

Definition (Complex numbers)

- Define basis elements 1 and i;
- Define complex numbers as a vector space over reals: elements have the form $x + iy$;
- One more axiom required: $i^2 = -1$.

Definition (Quaternions)

- Define basis elements 1, i, j, k;
- Define quaternions as a vector space over reals: elements have the form $q_0 + q_1 i + q_2 j + q_3 k$;
- One more axiom:

$$i^2 = j^2 = k^2 = ijk = -1$$

Basis element multiplication

From that one axiom, we can derive other products:

$$
\begin{aligned}
ijk &= -1 \\
i(ijk) &= i(-1) \\
-jk &= -i \\
jk &= i
\end{aligned}
$$

Writing them all down:

$$
\begin{aligned}
ij = k&, ji = -k \\
jk = i&, kj = -i \\
ki = j&, ik = -j
\end{aligned}
$$

Quaternion products of i, j, k behave like cross product.

Quaternion notation

We can write a quaternion several ways:

$$q = q_0 + q_1 i + q_2 j + q_3 k$$
$$q = (q_0, q_1, q_2, q_3)$$
$$q = q_0 + \mathbf{q}$$

Definition (Scalar part; vector part)

For quaternion $q_0 + \mathbf{q}$, q_0 is the scalar part and $\mathbf{q}$ is the vector part

Quaternion product

We can write a quaternion product several ways:

$$
\begin{aligned}
pq &= (p_0 + p_1 i + p_2 j + p_3 k)(q_0 + q_1 i + q_2 j + q_3 k) \\
&= (p_0 q_0 - p_1 q_1 - p_2 q_2 - p_3 q_3) + \ldots i + \ldots j + \ldots k \\
pq &= (p_0 + \mathbf{p})(q_0 + \mathbf{q}) \\
&= (p_0 q_0 + p_0 \mathbf{q} + q_0 \mathbf{p} + \mathbf{pq})
\end{aligned}
$$

The last product includes many different kinds of product: product of two reals, scalar product of vectors. But what is $\mathbf{pq}$? Cross product? Dot product? Both! Cross product minus dot product!

$$pq = (p_0 q_0 - \mathbf{p} \cdot \mathbf{q} + p_0 \mathbf{q} + q_0 \mathbf{p} + \mathbf{p} \times \mathbf{q})$$

Conjugate, length

Definition (Conjugate)

$$q^* = q_0 - q_1 i - q_2 j - q_3 k$$

Note that

$$\begin{aligned} qq^* &= (q_0 + \mathbf{q})(q_0 - \mathbf{q}) \\ &= q_0^2 + q_0\mathbf{q} - q_0\mathbf{q} - \mathbf{q}\mathbf{q} \\ &= q_0^2 + \mathbf{q} \cdot \mathbf{q} - \mathbf{q} \times \mathbf{q} \\ &= q_0^2 + q_1^2 + q_2^2 + q_3^2 \end{aligned}$$

Definition (Length)

$$|q| = \sqrt{qq^*} = \sqrt{q_0^2 + q_1^2 + q_2^2 + q_3^2}$$

Quaternion inverse

Every quaternion except 0 has an inverse:

$$q^{-1} = \frac{q^*}{|q|^2}$$

NOT COVERED

Without commutativity, quaternions are a *division ring*, or a *non-commutative field*, or a *skew field*.
Just as complex numbers are an extension of the reals, quaternions are an extension of the complex numbers (and of the reals).
If 1D numbers are the reals, and 2D numbers are the complex numbers, then 4D numbers are quaternions, and that's all there is. (Frobenius) (Octonions are not associative.)

Rotation using unit quaternions

- Let q be a unit quaternion, i.e. $|q| = 1$.
  - It can be expressed as

$$q = \cos\frac{\theta}{2} + \sin\frac{\theta}{2}\hat{\mathbf{n}}$$

- Let $x = 0 + \mathbf{x}$ be a "pure vector".
- Let $x' = qxq^*$.
- Then x' is the pure vector $\text{rot}(\theta, \hat{\mathbf{n}})\mathbf{x}$!!!

Proof that unit quaternions work

- Expand the product qxq^*;
- Apply half angle formulas;
- Simplify;
- Compare with Rodrigues's formula.

Sadly, not all proofs confer insight.

Why $\theta/2$? Why qxq^* instead of qx?

NOT COVERED

In analogy with complex numbers, why not use

$$p = \cos\theta + \hat{\mathbf{n}} \sin\theta$$
$$\mathbf{x}' = p\mathbf{x}$$

To explore that idea, define a map $L_p(q) = pq$. Note that $L_p(q)$ can be written:

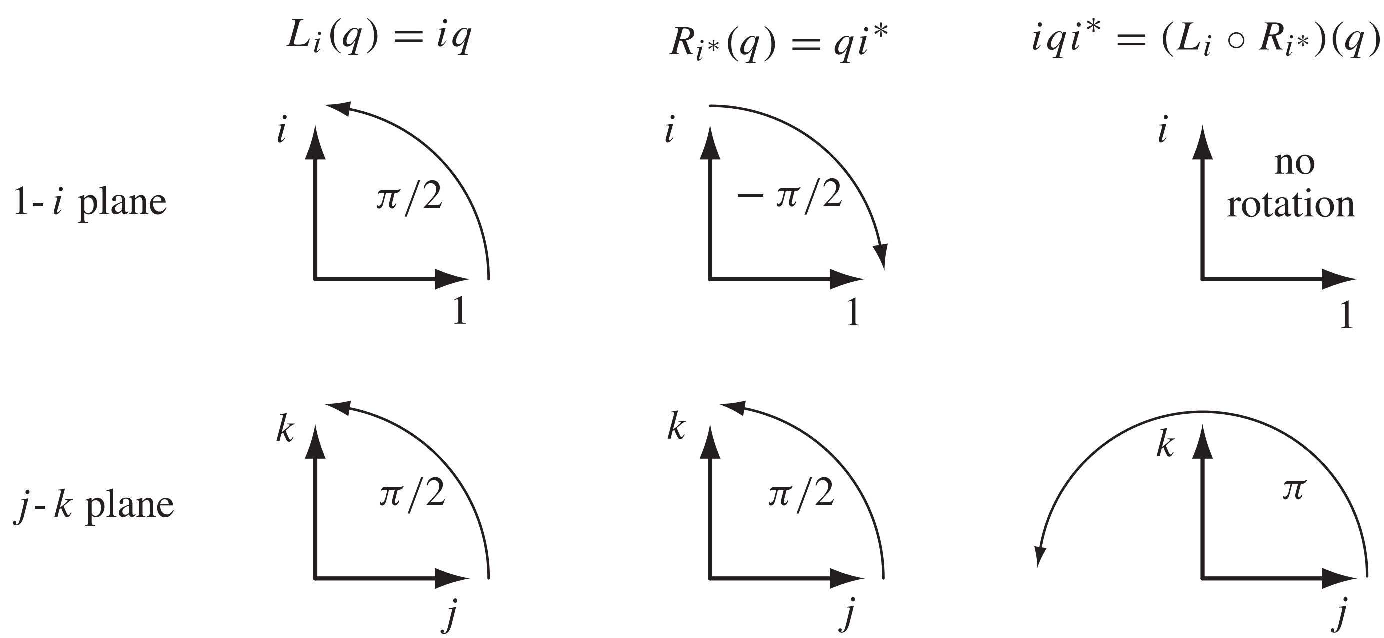
$$L_p(q) = \begin{pmatrix} p_0 & -p_1 & -p_2 & -p_3 \\ p_1 & p_0 & -p_3 & p_2 \\ p_2 & p_3 & p_0 & -p_1 \\ p_3 & -p_2 & p_1 & p_0 \end{pmatrix} \begin{pmatrix} q_0 \\ q_1 \\ q_2 \\ q_3 \end{pmatrix}$$

Note that the matrix above is orthonormal. L_p is a rotation of Euclidean 4 space!

Geometrical explanation
NOT COVERED

Although $L_p(q)$ rotates the 4D space of quaternions, it is *not* a rotation of the 3D subspace of pure vectors. Some of the 3D subspace leaks into the fourth dimension.

Consider an example using $p = i$. Is it a rotation about i of $\pi/2$?



What do we do with a representation?

Rotate a point: qxq^*.

Compose two rotations:

$$q(p\mathbf{x}p^*)q^* = (qp)\mathbf{x}(qp)^*$$

Convert to other representations:

- From axis-angle to quaternion:

$$q = \cos\frac{\theta}{2} + \sin\frac{\theta}{2}\hat{\mathbf{n}}$$

- From quaternion to axis-angle:

$$\theta = 2\tan^{-1}(|\mathbf{q}|, q_0)$$
$$\hat{\mathbf{n}} = \mathbf{q}/|\mathbf{q}|$$

assuming θ is nonzero.

From quaternion to rotation matrix

Just expand the product

$$qxq^* =$$
$$\begin{pmatrix} q_0^2+q_1^2-q_2^2-q_3^2 & 2(q_1q_2-q_0q_3) & 2(q_1q_3+q_0q_2) \\ 2(q_1q_2+q_0q_3) & q_0^2-q_1^2+q_2^2-q_3^2 & 2(q_2q_3-q_0q_1) \\ 2(q_1q_3-q_0q_2) & 2(q_2q_3+q_0q_1) & q_0^2-q_1^2-q_2^2+q_3^2 \end{pmatrix}\mathbf{x}$$