Unit 2: Locomotion
Kinematics of Wheeled Robots: Part 1

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1 Introduction

- Kinematics?
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- Representation

2 Wheel Kinematic Constraints

- Fixed Standard Wheel
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Kinematics? 

- *Kinematics* is the study of the motion of objects without being concerned with forces
  - *Dynamics* addresses forces
- Understanding a robot’s kinematics allows us to...
  - Know how a robot’s body constrains its motion
  - Determine the paths and trajectories a robot can achieve
- We will address the kinematics of wheeled robots
  - Far simpler than the kinematics of legged robots
- Each wheel allows motion in some direction(s) and *constrains* it in others
Notation

- Differs somewhat from book
- Scalars in both cases: e.g. $a, B, \beta, \dot{x}_R$
- Vectors in bold lower-case: e.g. $\mathbf{x}, \dot{\xi}$
- Matrices in bold upper-case: e.g. $A, R_{ccw}(\theta), \Phi(t)$
- Points and axes in upper-case: e.g. $P, X_R, Y_R$
We will consider only motion within the plane (i.e. 2D motion)

The *inertial* reference frame is the global coordinate system with origin $O$
- Denoted by subscript $I$
- Axes $X_I$ and $Y_I$

The robot’s reference frame has some origin $P$ and describes the layout of its body w.r.t. $P$
- (w.r.t. = “with respect to”)
- Denoted by subscript $R$
- Axes $X_R$ and $Y_R$
The robot’s pose, $\xi_l$, gives the position of $P$, and the orientation of the robot’s reference frame w.r.t. the inertial reference frame

$$
\xi_l = \begin{bmatrix} x \\ y \\ \theta \end{bmatrix}
$$
Assume we want the robot to travel with some velocity w.r.t. the global reference frame

\[ \dot{x}_I = \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \end{bmatrix} \]

where \( \dot{\theta} \) causes the robot to spin about \( P \)

We must determine how this motion vector is expressed w.r.t. the robot reference frame by rotating it back (i.e. clockwise) by \( \theta \)
Counterclockwise rotation in 2D is achieved by matrix multiplication with the rotation matrix $R_{ccw}(\theta)$:

$$R_{ccw}(\theta) = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}$$

(See supplementary notes on “The Rotation Matrix”)

However, we desire a clockwise rotation: $R_{cw}(\theta) = R_{ccw}(-\theta)$; Also, we wish to apply the rotation to $\xi_I$ which includes a third term, $\theta$; This term does not need to be modified by $R_{cw}(\theta)$ so we expand $R_{cw}(\theta)$ to leave it alone

$$R_{cw}(\theta) = \begin{bmatrix} \cos \theta & \sin \theta & 0 \\ -\sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$
Thus, the motion vector in the robot reference frame is given by

$$\dot{\xi}_R = R_{cw}(\theta)\dot{\xi}_I$$

(1)

We will refer to the individual components of $\dot{\xi}_R$ as

$$\dot{\xi}_R = \begin{bmatrix} \dot{x}_R \\ \dot{y}_R \\ \dot{\theta}_R \end{bmatrix}$$

Note that $\dot{\theta}_R = \dot{\theta}$; both describe rotation about the robot origin $P$. 
Example: A robot which is aimed down the negative $X_I$ axis is compelled to move in the direction $\dot{\xi}_I = [1 \ 0 \ 0]^T$. $\theta = \pi$, therefore

$$\dot{\xi}_R = R_{cw}(\theta)\dot{\xi}_I = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}$$

The robot should travel along the negative $X_R$ axis.
A forward kinematic model translates the robot’s actions into a motion vector in the inertial reference frame.

For most robots, an ‘action’ corresponds to setting the spin speeds of its wheels; e.g. spin speeds $\dot{\phi}_1$ and $\dot{\phi}_2$ for a 2-wheeled robot.

**Forward Kinematic Model**

$$\dot{\xi}_I = f(\dot{\phi}_1, \dot{\phi}_2)$$

An inverse kinematic model translates a desired motion into robot actions.

Can be difficult to obtain for some robots, e.g. manipulators.

Many ways to achieve some poses, no solution for others.

**Inverse Kinematic Model**

$$\begin{bmatrix} \dot{\phi}_1 \\ \dot{\phi}_2 \end{bmatrix}^T = g(\dot{\xi}_I)$$
Wheel Kinematic Constraints

- **Assumptions:**
  - Plane of wheel vertical to ground
  - Single point of contact
  - No sliding or skidding

- **Two Constraints:**
  - Rolling constraint:
    - Motion in wheel direction = Roll speed
  - Sliding constraint:
    - Component of motion orthogonal to wheel direction = 0
  - (wheel direction ≡ direction perpendicular to wheel axle)
Fixed Standard Wheel

- An unsteered wheel with a fixed position and angle w.r.t. chassis
- Located at distance $l$ and angle $\alpha$ from $P$
- Angle of wheel w.r.t. the ray from $P$ to wheel centre: $\beta$

![Diagram of a fixed standard wheel and its parameters.](image)

*Figure 3.4*
A fixed standard wheel and its parameters.
The wheel has radius \( r \) with rotational position \( \phi \)

The distance that the wheel has rolled is \( r\phi \)

- arc length = radius \(*\) angle

The roll speed is \( r \phi \)
The Rolling Constraint

The rolling constraint for a *fixed standard wheel* (FSW) is as follows:

\[
\sin(\alpha + \beta) - \cos(\alpha + \beta) (-l) \cos(\beta) \dot{\xi}_R = r \dot{\phi}
\]

This can be interpreted as a dot product between two vectors:

\[
\sin(\alpha + \beta) - \cos(\alpha + \beta) (-l) \cos(\beta)
\]

which describes the direction of the wheel and

\[\dot{\xi}_R\]

which is the velocity vector. Taking the dot product means finding the component of the velocity vector which is aligned with the wheel. This dot product must equal \(r \dot{\phi}\) which is the roll speed.

This constraint doesn’t say anything about motion orthogonal to the wheel. That’s the job of the sliding constraint...
The Sliding Constraint

The sliding constraint for a FSW is as follows:

\[
\begin{bmatrix}
\cos(\alpha + \beta) & \sin(\alpha + \beta) & l \sin(\beta)
\end{bmatrix}
\dot{\xi}_R = 0
\]

This equation dictates that there can be no motion in the direction orthogonal to the wheel.

See supplementary notes on “The Rolling and Sliding Constraints” for derivations of both.
Example

- Constraints on a robot with one FSW
- COVERED ON BOARD
Constraints for Other Wheel Types

- Steered standard wheel: Differs from FSW only in that steering angle is a function of time: $\beta(t)$
  - Rolling and sliding constraints same as for FSW. Why? Constraints are instantaneous, whereas impact of changes in steering angle must be integrated over time

- Castor wheel: Given any velocity vector, we can find a roll speed and steering angle to accommodate [See book for details]

- Swedish and spherical wheels: Can accommodate any velocity vector [See book]