Computer Science 4766/6912: Formula Sheet for Final Exam

Trigonometry

θ (radians)	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
θ (degrees)	$0°$	$30°$	$45°$	$60°$	$90°$
$\cos(\theta)$	1	$\sqrt{3}/2$	$\sqrt{2}/2$	$1/2$	0
$\sin(\theta)$	0	$1/2$	$\sqrt{2}/2$	$\sqrt{3}/2$	1

$$\sin(-\theta) = -\sin(\theta) \qquad \cos(-\theta) = \cos(\theta)$$
$$\sin(\theta - \tfrac{\pi}{2}) = -\cos(\theta) \qquad \cos(\theta - \tfrac{\pi}{2}) = \sin(\theta)$$

$$\begin{aligned} \cos(a+b) &= \cos a \cos b - \sin a \sin b \\ \sin(a+b) &= \sin a \cos b + \cos a \sin b \end{aligned}$$

Kinematics

The clockwise rotation matrix:

$$\boldsymbol{R}(\theta) = \boldsymbol{R_{cw}}(\theta) = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Relation between motion in the inertial and robot reference frames:

$$\dot{\boldsymbol{\xi}}_{\boldsymbol{R}} = \boldsymbol{R_{cw}}(\theta)\dot{\boldsymbol{\xi}}_{\boldsymbol{I}}$$

Rolling constraint for a fixed standard wheel:

$$[\sin(\alpha+\beta) \;\; -\cos(\alpha+\beta) \;\; (-l)\cos(\beta)]\, \dot{\boldsymbol{\xi}}_{\boldsymbol{R}} = r\dot{\boldsymbol{\phi}}$$

Sliding constraint for a fixed standard wheel:

$$[\cos(\alpha+\beta) \;\; \sin(\alpha+\beta) \;\; l\sin(\beta)]\, \dot{\boldsymbol{\xi}}_{\boldsymbol{R}} = \boldsymbol{0}$$

Spatial Transforms:

$${}^{A}P = {}^{A}_{B}R\,{}^{B}P + {}^{A}P_{BORG}$$

$$\begin{bmatrix} {}^{A}P \\ 1 \end{bmatrix} = \left[\begin{array}{c|c} {}^{A}_{B}R & {}^{A}P_{BORG} \\ \hline 0 \;\; 0 \;\; 0 & 1 \end{array}\right] \begin{bmatrix} {}^{B}P \\ 1 \end{bmatrix}$$

Quaternion axioms:

$$i^2 = j^2 = k^2 = ijk = -1$$

$ij = k$, $ji = -k$, $jk = i$, $kj = -i$, $ki = j$, $ik = -j$

Conversion from axis-angle representation to unit quaternion:

$$q = \cos\frac{\theta}{2} + \sin\frac{\theta}{2}\boldsymbol{n}$$

Calculus:

$\frac{d}{dx}(\sin x) = \cos x \qquad \frac{d}{dx}(\cos x) = -\sin x$

$\frac{d}{dx}(\tan x) = \frac{1}{\cos^2 x} \qquad \frac{d}{dx}(\arctan x) = \frac{1}{1+x^2}$

Product Rule: $(fg)' = f'g + fg'$

Quotient Rule: $(\frac{f}{g})' = \frac{f'g - fg'}{g^2}$

Jacobian Matrix: Input $\boldsymbol{x}$, Output $\boldsymbol{y}$

$$\begin{bmatrix} \frac{\partial y_1}{\partial x_1} & \cdots & \frac{\partial y_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial y_m}{\partial x_1} & \cdots & \frac{\partial y_m}{\partial x_n} \end{bmatrix}$$

Probability and Statistics:

Conditional Probability:

$$p(A|B) = \frac{p(A \wedge B)}{p(B)}$$

Bayes Rule:

$$p(A|B) = \frac{p(B|A)p(A)}{p(B)}$$

Theorem of Total Probability for mutually exclusive and exhaustive events $A_1, \ldots, A_n$:

$$p(B) = \sum_{i=1}^{n} P(B|A_i)P(A_i)$$

Two events A and B are independent iff:

$$p(A \wedge B) = p(A)p(B)$$

Two events A and B are conditionally independent given C iff:

$$p(A \wedge B|C) = p(A|C)p(B|C)$$

Covariance:

$$\sigma_{ij} = Cov(X_i, X_j) = E((X_i - \mu_{x_i})(X_j - \mu_{x_j}))$$

A single random variable passes through a linear function $y = f(x) = a \cdot x + b$

$$\mu_y = a \cdot \mu_x + b, \qquad \sigma_y^2 = a^2 \cdot \sigma_x^2$$

A **vector** of random variables passes through a linear function $\boldsymbol{y} = A\boldsymbol{x}$

$$\mu_{\boldsymbol{y}} = A\mu_{\boldsymbol{x}}, \qquad \Sigma_{\boldsymbol{y}} = A\Sigma_{\boldsymbol{x}}A^T$$

where $\Sigma_{\boldsymbol{x}}$ is the **covariance matrix**,

$$\begin{aligned} \Sigma_{\boldsymbol{x}} &= E\left[(\boldsymbol{x} - E[\boldsymbol{x}])(\boldsymbol{x} - E[\boldsymbol{x}])^T\right] \\ &= \begin{bmatrix} \sigma_{x_1}^2 & \sigma_{x_1,x_2} & \cdots & \sigma_{x_1,x_n} \\ \sigma_{x_2,x_1} & \sigma_{x_2}^2 & \cdots & \sigma_{x_2,x_n} \\ \vdots & \vdots & & \vdots \\ \sigma_{x_n,x_1} & \sigma_{x_n,x_2} & \cdots & \sigma_{x_n}^2 \end{bmatrix} \end{aligned}$$

Bayes Filter:

For all x_t,

$$
\begin{aligned}
\overline{bel}(x_t) &= \int p(x_t|u_t, x_{t-1}) bel(x_{t-1}) dx_{t-1} \\
bel(x_t) &= \eta p(z_t|x_t)\overline{bel}(x_t)
\end{aligned}
$$

Odometry motion model parameters:

$$
\begin{aligned}
\delta_{rot1} &= \text{atan2}(\bar{y}' - \bar{y}, \bar{x}' - \bar{x}) - \bar{\theta} \\
\delta_{trans} &= \sqrt{(\bar{x} - \bar{x}')^2 + (\bar{y} - \bar{y}')^2} \\
\delta_{rot2} &= \bar{\theta}' - \bar{\theta} - \delta_{rot1}
\end{aligned}
$$

Particle Filter Algorithm

1. Particle_Filter(χ_{t-1}, u_t, z_t)
2. $\overline{\chi}_t = \chi_t = \emptyset$
3. for $m = 1$ to M do
4. sample $x_t^{[m]} \sim p(x_t|u_t, x_{t-1}^{[m]})$
5. $w_t^{[m]} = p(z_t|x_t^{[m]})$
6. $\overline{\chi}_t = \overline{\chi}_t + \left\langle x_t^{[m]}, w_t^{[m]} \right\rangle$
7. endfor
8. for $m = 1$ to M do
9. draw i with probability $\propto w_t^{[i]}$
10. add $x_t^{[i]}$ to χ_t
11. endfor
12. return χ_t

Kalman Filter:

1. Kalman_Filter($\mu_{t-1}, \Sigma_{t-1}, u_t, z_t$)
2. $\bar{\mu}_t = A_t \mu_{t-1} + B_t u_t$
3. $\bar{\Sigma}_t = A_t \Sigma_{t-1} A_t^T + R_t$
4. $K_t = \bar{\Sigma}_t C_t^T (C_t \bar{\Sigma}_t C_t^T + Q_t)^{-1}$
5. $\mu_t = \bar{\mu}_t + K_t(z_t - C_t \bar{\mu}_t)$
6. $\Sigma_t = (I - K_t C_t)\bar{\Sigma}_t$
7. return μ_t, Σ_t

The Extended Kalman Filter:

1. EKF($\mu_{t-1}, \Sigma_{t-1}, u_t, z_t$)
2. $\bar{\mu}_t = g(u_t, \mu_{t-1})$
3. $\bar{\Sigma}_t = G_t \Sigma_{t-1} G_t^T + R_t$
4. $K_t = \bar{\Sigma}_t H_t^T (H_t \bar{\Sigma}_t H_t^T + Q_t)^{-1}$
5. $\mu_t = \bar{\mu}_t + K_t(z_t - h(\bar{\mu}_t))$
6. $\Sigma_t = (I - K_t H_t)\bar{\Sigma}_t$
7. return μ_t, Σ_t

Value Iteration (Version 2):

```
discreteValueIteration()
    for i = 1 to N
        V(x_i) = r_min
    end for
    repeat until convergence
        for i = 1 to N
            V(x_i) = γ max_u [ Σ_{j=1}^{N} p(x_j|u, x_i) (r(x_j, u) + V(x_j)) ]
        end for
    end repeat

controlPolicy(x_i, V)
    return argmax_u [ Σ_{j=1}^{N} p(x_j|u, x_i) (r(x_j, u) + V(x_j)) ]
```

$$V(x_i) = \gamma \max_u \left[\sum_{j=1}^{N} p(x_j|u, x_i) \left(r(x_j, u) + V(x_j) \right) \right]$$

$$\operatorname*{argmax}_u \left[\sum_{j=1}^{N} p(x_j|u, x_i) \left(r(x_j, u) + V(x_j) \right) \right]$$