Localization: Part 5
Covariance Matrices and the Multivariate Normal Distribution

Computer Science 4766/6912

Department of Computer Science
Memorial University of Newfoundland

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Enough about covariance for now... What about covariance matrices?
They come up more naturally if we look at functions of random variables...

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- What if we have multiple outputs? Things get complicated and it becomes helpful to introduce the...

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Covariance matrices have a number of special properties: square, symmetric, positive semidefinite

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- The variance σ^2 is now the covariance matrix Σ, which is a square matrix of size $n \times n$

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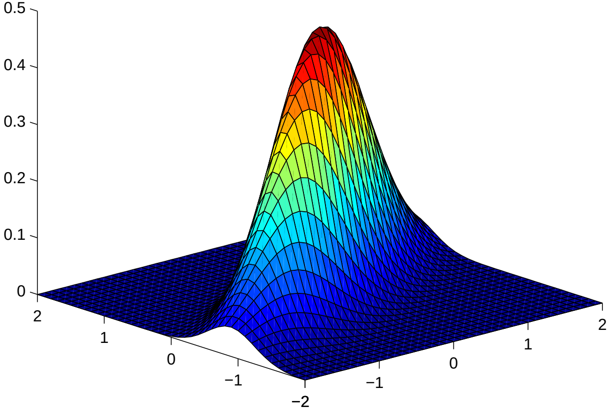

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For some values of x we get the same probability; In particular, when the expression $(x-\mu)^T\Sigma^{-1}(x-\mu)$ is constant we get a curve of constant probability; What curve? Define a constant c^2 as follows (shown in 2-D but works in higher dimensions as well),

$$[x_1, y_1]\begin{bmatrix} a & b \\ b & d \end{bmatrix}\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = c^2$$

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which is the equation of an ellipse.

It can be shown that the probability of a value falling within the ellipse is $1 - \alpha$, where α comes from $\chi^2(\alpha) = c^2$ and $\chi^2(\alpha)$ is the upper (100α)th percentile of a χ^2 distribution.

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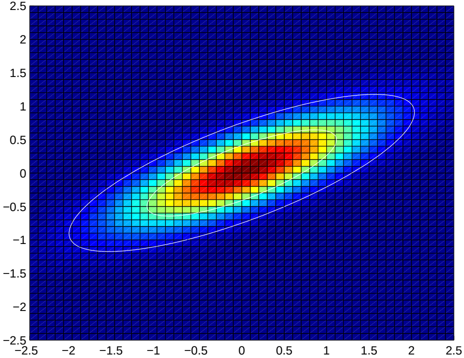

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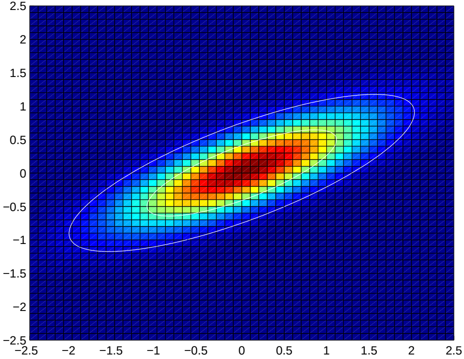

The smaller ellipse is the 50% confidence ellipse, while the larger is the 90% confidence ellipse