Localization: Part 5
Covariance Matrices and the Multivariate Normal Distribution

Computer Science 4766/6912

Department of Computer Science
Memorial University of Newfoundland

July 26, 2017
To discuss the Kalman filter, we first need to discuss the **multivariate normal distribution**...
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To discuss covariance matrices, we first need to *review covariance*...
The effect of one random variable on another is expressed by the **covariance** between the two variables.
Covariance

The effect of one random variable on another is expressed by the **covariance** between the two variables; This is defined as,

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\sigma_{ij} = \text{Cov}(X_i, X_j) = \mathbb{E}((X_i - \mu_{X_i})(X_j - \mu_{X_j})) = \mathbb{E}(X_i X_j) - \mu_{X_i} \mu_{X_j}
\]

Unlike variance, covariance can be positive or negative:

- Positive covariance implies that large values of \(X_i\) are associated with large values of \(X_j\)
- Negative covariance implies that large values of \(X_i\) are associated with small values of \(X_j\), and vice versa

Related to covariance, is the **correlation coefficient**, which indicates a linear relationship between two random variables,

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Enough about covariance for now... What about covariance matrices? They come up more naturally if we look at functions of random variables...
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Functions of Random Variables

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Let \( y = f(x) = ax + b \)

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- What if we have multiple outputs? Things get complicated and it becomes helpful to introduce the...
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(Note that the symbol $\Sigma$ will be used to represent covariance matrices. It should be clear from the context whether a covariance matrix or a summation operator is intended.)

We now want to know $\Sigma_y$ for our system; It turns out that this is given by, 

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\vdots & \vdots & \ddots & \vdots \\
\sigma_{x,n},x_1 & \sigma_{x,n},x_2 & \cdots & \sigma^2_{x,n}
\end{bmatrix} \]

The formal definition of a covariance matrix is equivalent to the definition of covariance extended to vectors,

\[ \Sigma = \mathbb{E}\left[ (x - \mathbb{E}[x])(x - \mathbb{E}[x])^T \right] \]

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- The mean \( \mu \) is now a vector of size \( n \)
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\[ p(x) = (2\pi\sigma^2)^{-\frac{1}{2}} \exp \left\{ -\frac{1}{2} \frac{(x - \mu)^2}{\sigma^2} \right\} \]

- If \( x \) is a \( n \)-dimensional vector then the multivariate Normal p.d.f. is

\[ p(x) = \det (2\pi\Sigma)^{-\frac{1}{2}} \exp \left\{ -\frac{1}{2} (x - \mu)^T \Sigma^{-1} (x - \mu) \right\} \]

- The mean \( \mu \) is now a vector of size \( n \)
- The variance \( \sigma^2 \) is now the covariance matrix \( \Sigma \), which is a square matrix of size \( n \times n \)
The following is a 2-D normal distribution with \( \mu = [0, 0]^T \) and
\[
\Sigma = \begin{bmatrix}
0.9 & 0.4 \\
0.4 & 0.3
\end{bmatrix}
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$$\Sigma = \begin{bmatrix} 0.9 & 0.4 \\ 0.4 & 0.3 \end{bmatrix}$$
Consider again the definition of the multivariate Normal,

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\[
\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = c^2
\]

\[
\begin{pmatrix} a & b \\ b & d \end{pmatrix}
\]

\[
\begin{pmatrix} x_1 \\ y_1 \end{pmatrix}^T = (x - \mu) \quad \text{and the matrix in the middle is } \Sigma^{-1}.
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This can be expanded to obtain,

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a x_1^2 + 2 b x_1 y_1 + d y_1^2 = c^2
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which is the equation of an ellipse.
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which is the equation of an ellipse.
It can be shown that the probability of a value falling within the ellipse is $1 - \alpha$, where $\alpha$ comes from $\chi^2(\alpha) = c^2$ and $\chi^2(\alpha)$ is the upper $(100\alpha)$th percentile of a $\chi^2$ distribution.
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An ellipse with \(\alpha = 0.1\) is known as a 90\% confidence ellipse,
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The smaller ellipse is the 50% confidence ellipse, while the larger is the 90% confidence ellipse.