Supplementary Material:
The Rolling and Sliding Constraints

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We derive here the rolling and sliding constraints for a fixed standard wheel; First the rolling constraint...

We wish to express the following more formally,

- Component of motion in wheel direction = Roll speed

"Roll speed" comes from differentiating "roll position" w.r.t. time

- Wheel radius, \( r \); rotation, \( \phi \)
- Roll position = \( r\phi \)
- Roll speed = \( r\dot{\phi} \)

Next we differentiate the position of the wheel to obtain the wheel motion,

\[
\mathbf{w}_I = \begin{bmatrix} x + l \cos(\theta + \alpha) \\ y + l \sin(\theta + \alpha) \end{bmatrix}
\]

\[
\dot{\mathbf{w}}_I = \begin{bmatrix} \dot{x} - l\dot{\theta} \sin(\theta + \alpha) \\ \dot{y} + l\dot{\theta} \cos(\theta + \alpha) \end{bmatrix}
\]

We will need the motion of the wheel in the robot ref. frame, \( \mathbf{w}_R \),

\[
\mathbf{w}_R = R(\theta)\mathbf{w}_I = \begin{bmatrix} \dot{x}_R + l\dot{\theta} \sin(\theta + \alpha) \\ \dot{y}_R + l\dot{\theta} \cos(\theta + \alpha) \end{bmatrix}
\]

\[
\mathbf{w}_R = \begin{bmatrix} x_{\mathbf{r}} \\ y_{\mathbf{r}} \end{bmatrix}
\]

\[= t + r \]

where \( t = [x_{\mathbf{r}}, y_{\mathbf{r}}]^T \) represents the motion of the wheel due to translation of the robot and \( r = l\dot{\theta}[-\sin \alpha, \cos \alpha]^T \) represents the motion due to the robot’s rotation.
The wheel’s forward direction is expressed by the unit vector $v$.

Finally, the “component of motion in the wheel direction” is obtained... by taking the dot product of $\dot{w}_R = t + r$ with $v$...

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Requires the trigonometric identities:

\[
\sin(x + y) = \sin x \cos y + \cos x \sin y \\
\cos(x + y) = \cos x \cos y - \sin x \sin y
\]

The Sliding Constraint

- We now wish to express the following more formally,
  - Component of motion orthogonal to wheel direction $= 0$
  - The derivation is quite similar to that of the rolling constraint, with the following change:
    - For the rolling constraint we had $v \cdot (r + t) = r \dot{\phi}$
    - Replace $v$ with $n$, orthogonal to $v$; Replace $r \dot{\phi}$ with 0

Expanding $n \cdot (r + t) = 0$ we arrive at the sliding constraint in its final form

\[
\begin{bmatrix}
\cos(\alpha + \beta) & \sin(\alpha + \beta) & l \sin \beta
\end{bmatrix} R(\theta) \dot{\xi}_I = 0
\]