Supplementary Material: The Rolling and Sliding Constraints

Computer Science 4766/6912

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May 23, 2018

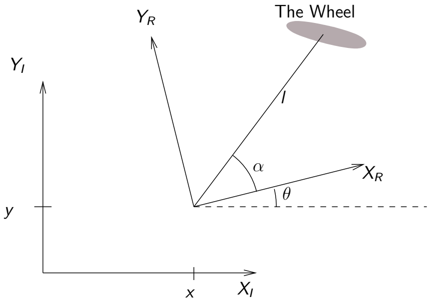
The Rolling Constraint

We derive here the rolling and sliding constraints for a fixed standard wheel; First the rolling constraint...

- We wish to express the following more formally,
  - Component of motion in wheel direction = Roll speed
- "Roll speed" comes from differentiating "roll position" w.r.t. time
  - Wheel radius, r; rotation, ϕ
  - Roll position $= r\phi$
  - Roll speed $= r\dot{\phi}$

“Component of **motion** in wheel direction”?

- First of all, what motion are we referring to?
- $\dot{\boldsymbol{\xi}}_{\boldsymbol{I}}$, the motion of the robot in the global ref. frame?
- $\dot{\boldsymbol{\xi}}_{\boldsymbol{R}}$, the motion of the robot in the robot ref. frame?
- No — we need the motion of the wheel itself in the robot ref. frame
- First, express the position of the wheel in the global ref. frame: $\boldsymbol{w_I}$



$$\boldsymbol{w_I} = \begin{bmatrix} x + l\cos(\theta+\alpha) \\ y + l\sin(\theta+\alpha) \end{bmatrix}$$

Next we differentiate the position of the wheel to obtain the wheel motion,

$$
\begin{aligned}
\boldsymbol{w_I} &= \begin{bmatrix} x + l\cos(\theta+\alpha) \\ y + l\sin(\theta+\alpha) \end{bmatrix} \\
\dot{\boldsymbol{w}}_{\boldsymbol{I}} &= \begin{bmatrix} \dot{x} - l\dot{\theta}\sin(\theta+\alpha) \\ \dot{y} + l\dot{\theta}\cos(\theta+\alpha) \end{bmatrix}
\end{aligned}
$$

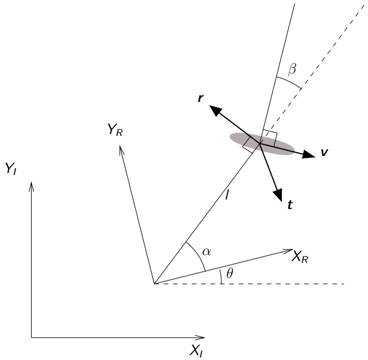
We will need the motion of the wheel in the robot ref. frame,

$$
\begin{aligned}
\dot{\boldsymbol{w}}_{\boldsymbol{R}} &= \boldsymbol{R}(\theta)\dot{\boldsymbol{w}}_{\boldsymbol{I}} \\
\dot{\boldsymbol{w}}_{\boldsymbol{R}} &= \boldsymbol{R}(\theta)\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} + \boldsymbol{R}(\theta)l\dot{\theta}\begin{bmatrix} -\sin(\theta+\alpha) \\ \cos(\theta+\alpha) \end{bmatrix} \\
\dot{\boldsymbol{w}}_{\boldsymbol{R}} &= \begin{bmatrix} \dot{x_R} \\ \dot{y_R} \end{bmatrix} + l\dot{\theta}\begin{bmatrix} -\sin\alpha \\ \cos\alpha \end{bmatrix} \\
&= \boldsymbol{t} + \boldsymbol{r}
\end{aligned}
$$

where $\boldsymbol{t} = [\dot{x_R}, \dot{y_R}]^T$ represents the motion of the wheel due to **translation** of the robot and $\boldsymbol{r} = l\dot{\theta}[-\sin\alpha, \cos\alpha]^T$ represents the motion due to the robot's **rotation**

$$
\begin{aligned}
\dot{\boldsymbol{w}}_{\boldsymbol{R}} &= \begin{bmatrix} \dot{x}_R \\ \dot{y}_R \end{bmatrix} + l\dot{\theta} \begin{bmatrix} -\sin\alpha \\ \cos\alpha \end{bmatrix} \\
&= \boldsymbol{t} + \boldsymbol{r}
\end{aligned}
$$

The wheel's forward direction is expressed by the unit vector $\boldsymbol{v}$



Note that $\boldsymbol{r}$ points orthogonally to the line from P to the wheel (why?); $\boldsymbol{t}$ could point in any direction

Finally, the "component of motion in the wheel direction" is obtained... by taking the dot product of $\dot{\boldsymbol{w}}_{\boldsymbol{R}} = \boldsymbol{t} + \boldsymbol{r}$ with $\boldsymbol{v}$...

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Requires the trigonometric identities:

$$\begin{aligned}\sin(x+y) &= \sin x \cos y + \cos x \sin y \\ \cos(x+y) &= \cos x \cos y - \sin x \sin y\end{aligned}$$

The Sliding Constraint

- We now wish to express the following more formally,
  - Component of motion orthogonal to wheel direction $= 0$
- The derivation is quite similar to that of the rolling constraint, with the following change:
  - For the rolling constraint we had $\boldsymbol{v} \cdot (\boldsymbol{r} + \boldsymbol{t}) = r\dot{\phi}$
  - Replace $\boldsymbol{v}$ with $\boldsymbol{n}$, orthogonal to $\boldsymbol{v}$; Replace $r\dot{\phi}$ with 0

Expanding $\boldsymbol{n} \cdot (\boldsymbol{r} + \boldsymbol{t}) = 0$ we arrive at the sliding constraint in its final form

$$[\cos(\alpha + \beta) \;\; \sin(\alpha + \beta) \;\; l \sin \beta] \; \boldsymbol{R}(\theta)\dot{\boldsymbol{\xi}}_{\boldsymbol{I}} = 0$$