Supplementary Material:
The Rolling and Sliding Constraints

Computer Science 4766/6912

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We derive here the rolling and sliding constraints for a fixed standard wheel.
The Rolling Constraint

We derive here the rolling and sliding constraints for a fixed standard wheel; First the rolling constraint...
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- We wish to express the following more formally,
We derive here the rolling and sliding constraints for a fixed standard wheel; First the rolling constraint...

- We wish to express the following more formally,
  - Component of motion in wheel direction $=$ Roll speed

\[ \text{Roll speed} = r \dot{\phi} \]
We derive here the rolling and sliding constraints for a fixed standard wheel; First the rolling constraint...

- We wish to express the following more formally,
  - Component of motion in wheel direction = Roll speed
  - “Roll speed” comes from differentiating “roll position” w.r.t. time
The Rolling Constraint

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- We wish to express the following more formally,
  - Component of motion in wheel direction = Roll speed
- “Roll speed” comes from differentiating “roll position” w.r.t. time
  - Wheel radius, $r$; rotation, $\phi$
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- We wish to express the following more formally,
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  - Roll position $= r\phi$
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- We wish to express the following more formally,
  - Component of motion in wheel direction = Roll speed
- “Roll speed” comes from differentiating “roll position” w.r.t. time
  - Wheel radius, $r$; rotation, $\phi$
  - Roll position $= r\phi$
  - Roll speed $= r\dot{\phi}$
“Component of \textbf{motion} in wheel direction”? 

- First of all, what motion are we referring to?
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- First of all, what motion are we referring to?
- \( \dot{\xi}_I \), the motion of the robot in the global ref. frame?
- \( \dot{\xi}_R \), the motion of the robot in the robot ref. frame?

\[
\text{The Wheel } w_I = \begin{bmatrix} x + l \cos(\theta + \alpha) \\ y + l \sin(\theta + \alpha) \end{bmatrix}
\]
“Component of motion in wheel direction”? 

- First of all, what motion are we referring to?
- \( \dot{\xi}_I \), the motion of the robot in the global ref. frame?
- \( \dot{\xi}_R \), the motion of the robot in the robot ref. frame?
- No — we need the motion of the wheel itself in the robot ref. frame.
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- $\dot{\xi}_R$, the motion of the robot in the robot ref. frame?
- No — we need the motion of the wheel itself in the robot ref. frame.
- First, express the position of the wheel in the global ref. frame: $\mathbf{w}_I$
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“Component of motion in wheel direction”? 

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- $\dot{\xi}_I$, the motion of the robot in the global ref. frame?
- $\dot{\xi}_R$, the motion of the robot in the robot ref. frame?
- No — we need the motion of the wheel itself in the robot ref. frame
- First, express the position of the wheel in the global ref. frame:  $\mathbf{w}_I$ 

$$\mathbf{w}_I = \begin{bmatrix} x + l \cos(\theta + \alpha) \\ y + l \sin(\theta + \alpha) \end{bmatrix}$$
Next we differentiate the position of the wheel to obtain the wheel motion,
Next we differentiate the position of the wheel to obtain the wheel motion,

\[
\mathbf{w}_I = \begin{bmatrix} x + l \cos(\theta + \alpha) \\ y + l \sin(\theta + \alpha) \end{bmatrix}
\]
Next we differentiate the position of the wheel to obtain the wheel motion,

\[
\begin{align*}
\mathbf{w}_I &= \begin{bmatrix}
x + l \cos(\theta + \alpha) \\
y + l \sin(\theta + \alpha)
\end{bmatrix} \\
\dot{\mathbf{w}}_I &= \begin{bmatrix}
\dot{x} - l \dot{\theta} \sin(\theta + \alpha) \\
\dot{y} + l \dot{\theta} \cos(\theta + \alpha)
\end{bmatrix}
\end{align*}
\]
Next we differentiate the position of the wheel to obtain the wheel motion,

\[
\begin{align*}
\mathbf{w}_I &= \begin{bmatrix} x + l \cos(\theta + \alpha) \\ y + l \sin(\theta + \alpha) \end{bmatrix} \\
\mathbf{\dot{w}}_I &= \begin{bmatrix} \dot{x} - l \dot{\theta} \sin(\theta + \alpha) \\ \dot{y} + l \dot{\theta} \cos(\theta + \alpha) \end{bmatrix}
\end{align*}
\]

We will need the motion of the wheel in the robot ref. frame,
Next we differentiate the position of the wheel to obtain the wheel motion,

\[
\mathbf{w}_I = \begin{bmatrix} x + l \cos(\theta + \alpha) \\ y + l \sin(\theta + \alpha) \end{bmatrix}
\]

\[
\dot{\mathbf{w}}_I = \begin{bmatrix} \dot{x} - l \dot{\theta} \sin(\theta + \alpha) \\ \dot{y} + l \dot{\theta} \cos(\theta + \alpha) \end{bmatrix}
\]

We will need the motion of the wheel in the robot ref. frame,

\[
\dot{\mathbf{w}}_R = R(\theta) \dot{\mathbf{w}}_I
\]
Next we differentiate the position of the wheel to obtain the wheel motion,

\[
\begin{align*}
\mathbf{w}_I &= \begin{bmatrix} x + l \cos(\theta + \alpha) \\ y + l \sin(\theta + \alpha) \end{bmatrix} \\
\dot{\mathbf{w}}_I &= \begin{bmatrix} \dot{x} - l \dot{\theta} \sin(\theta + \alpha) \\ \dot{y} + l \dot{\theta} \cos(\theta + \alpha) \end{bmatrix}
\end{align*}
\]

We will need the motion of the wheel in the robot ref. frame,

\[
\begin{align*}
\dot{\mathbf{w}}_R &= R(\theta) \dot{\mathbf{w}}_I \\
\dot{\dot{\mathbf{w}}}_R &= R(\theta) \begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} + R(\theta) l \dot{\theta} \begin{bmatrix} -\sin(\theta + \alpha) \\ \cos(\theta + \alpha) \end{bmatrix}
\end{align*}
\]
Next we differentiate the position of the wheel to obtain the wheel motion,

\[
\begin{align*}
  \mathbf{w}_I &= \begin{bmatrix} x + l \cos(\theta + \alpha) \\ y + l \sin(\theta + \alpha) \end{bmatrix} \\
  \dot{\mathbf{w}}_I &= \begin{bmatrix} \dot{x} - l \dot{\theta} \sin(\theta + \alpha) \\ \dot{y} + l \dot{\theta} \cos(\theta + \alpha) \end{bmatrix}
\end{align*}
\]

We will need the motion of the wheel in the robot ref. frame,

\[
\begin{align*}
  \dot{\mathbf{w}}_R &= R(\theta) \dot{\mathbf{w}}_I \\
  \dot{\mathbf{w}}_R &= R(\theta) \begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} + R(\theta) l \dot{\theta} \begin{bmatrix} -\sin(\theta + \alpha) \\ \cos(\theta + \alpha) \end{bmatrix} \\
  \dot{\mathbf{w}}_R &= \begin{bmatrix} \dot{x}_R \\ \dot{y}_R \end{bmatrix} + l \dot{\theta} \begin{bmatrix} -\sin \alpha \\ \cos \alpha \end{bmatrix}
\end{align*}
\]
Next we differentiate the position of the wheel to obtain the wheel motion,

\[
\begin{align*}
\mathbf{w}_I &= \begin{bmatrix} x + l \cos(\theta + \alpha) \\ y + l \sin(\theta + \alpha) \end{bmatrix} \\
\dot{\mathbf{w}}_I &= \begin{bmatrix} \dot{x} - l \dot{\theta} \sin(\theta + \alpha) \\ \dot{y} + l \dot{\theta} \cos(\theta + \alpha) \end{bmatrix}
\end{align*}
\]

We will need the motion of the wheel in the robot ref. frame,

\[
\begin{align*}
\dot{\mathbf{w}}_R &= R(\theta) \dot{\mathbf{w}}_I \\
\ddot{\mathbf{w}}_R &= R(\theta) \begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} + R(\theta) \dot{l} \dot{\theta} \begin{bmatrix} -\sin(\theta + \alpha) \\ \cos(\theta + \alpha) \end{bmatrix} \\
\dddot{\mathbf{w}}_R &= \begin{bmatrix} \ddot{x}_R \\ \ddot{y}_R \end{bmatrix} + l \ddot{\theta} \begin{bmatrix} -\sin \alpha \\ \cos \alpha \end{bmatrix} \\
&= \mathbf{t} + \mathbf{r}
\end{align*}
\]
Next we differentiate the position of the wheel to obtain the wheel motion,

\[
\begin{align*}
\mathbf{w}_I &= \begin{bmatrix}
x + l \cos(\theta + \alpha) \\
y + l \sin(\theta + \alpha)
\end{bmatrix} \\
\dot{\mathbf{w}}_I &= \begin{bmatrix}
\dot{x} - l \dot{\theta} \sin(\theta + \alpha) \\
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\end{bmatrix}
\end{align*}
\]

We will need the motion of the wheel in the robot ref. frame,

\[
\begin{align*}
\dot{\mathbf{w}}_R &= R(\theta) \dot{\mathbf{w}}_I \\
\dot{\mathbf{w}}_R &= R(\theta) \begin{bmatrix}
\dot{x} \\
\dot{y}
\end{bmatrix} + R(\theta) l \dot{\theta} \begin{bmatrix}
-\sin(\theta + \alpha) \\
\cos(\theta + \alpha)
\end{bmatrix} \\
\dot{\mathbf{w}}_R &= \begin{bmatrix}
\dot{x}_R \\
\dot{y}_R
\end{bmatrix} + l \dot{\theta} \begin{bmatrix}
-\sin \alpha \\
\cos \alpha
\end{bmatrix} \\
&= \mathbf{t} + \mathbf{r}
\end{align*}
\]

where \( \mathbf{t} = [\dot{x}_R, \dot{y}_R]^T \) represents the motion of the wheel due to translation of the robot.
Next we differentiate the position of the wheel to obtain the wheel motion,

$$\mathbf{w}_I = \begin{bmatrix} x + l \cos(\theta + \alpha) \\ y + l \sin(\theta + \alpha) \end{bmatrix}$$

$$\dot{\mathbf{w}}_I = \begin{bmatrix} \dot{x} - l \dot{\theta} \sin(\theta + \alpha) \\ \dot{y} + l \dot{\theta} \cos(\theta + \alpha) \end{bmatrix}$$

We will need the motion of the wheel in the robot ref. frame,

$$\dot{\mathbf{w}}_R = R(\theta) \dot{\mathbf{w}}_I$$

$$\dot{\mathbf{w}}_R = R(\theta) \begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} + R(\theta) l \dot{\theta} \begin{bmatrix} - \sin(\theta + \alpha) \\ \cos(\theta + \alpha) \end{bmatrix}$$

$$\dot{\mathbf{w}}_R = \begin{bmatrix} \dot{x}_R \\ \dot{y}_R \end{bmatrix} + l \dot{\theta} \begin{bmatrix} - \sin \alpha \\ \cos \alpha \end{bmatrix} = \mathbf{t} + \mathbf{r}$$

where $\mathbf{t} = [\dot{x}_R, \dot{y}_R]^T$ represents the motion of the wheel due to translation of the robot and $\mathbf{r} = l \dot{\theta} [-\sin \alpha, \cos \alpha]^T$ represents the motion due to the robot's rotation.
\[
\begin{align*}
\dot{\mathbf{w}}_R &= \left[ \begin{array}{c} \dot{x}_R \\ \dot{y}_R \end{array} \right] + l \dot{\theta} \left[ \begin{array}{c} -\sin \alpha \\ \cos \alpha \end{array} \right] \\
&= \mathbf{t} + \mathbf{r}
\end{align*}
\]
\[
\begin{align*}
\dot{w}_R &= \begin{bmatrix} \dot{x}_R \\ \dot{y}_R \end{bmatrix} + l\dot{\theta} \begin{bmatrix} -\sin \alpha \\ \cos \alpha \end{bmatrix} \\
&= t + r
\end{align*}
\]

The wheel’s forward direction is expressed by the unit vector \( \mathbf{v} \).
\[ \dot{w}_R = \begin{bmatrix} \dot{x}_R \\ \dot{y}_R \end{bmatrix} + l \dot{\theta} \begin{bmatrix} -\sin \alpha \\ \cos \alpha \end{bmatrix} = t + r \]

The wheel’s forward direction is expressed by the unit vector \( v \).
\[
\dot{\mathbf{w}}_R = \begin{bmatrix}
\dot{x}_R \\
\dot{y}_R
\end{bmatrix} + l\dot{\theta} \begin{bmatrix} -\sin \alpha \\
\cos \alpha
\end{bmatrix} = \mathbf{t} + \mathbf{r}
\]

The wheel’s forward direction is expressed by the unit vector \( \mathbf{v} \)

Note that \( \mathbf{r} \) points orthogonally to the line from \( P \) to the wheel (why?)
\[ \dot{w}_R = \begin{bmatrix} x_R' \\ y_R' \end{bmatrix} + l\dot{\theta} \begin{bmatrix} -\sin \alpha \\ \cos \alpha \end{bmatrix} = t + r \]

The wheel’s forward direction is expressed by the unit vector \( \mathbf{v} \)

Note that \( r \) points orthogonally to the line from \( P \) to the wheel (why?); \( t \) could point in any direction
\[ \dot{\omega}_R = \begin{bmatrix} \dot{x}_R \\ \dot{y}_R \end{bmatrix} + l\dot{\theta} \begin{bmatrix} -\sin \alpha \\ \cos \alpha \end{bmatrix} = \mathbf{t} + \mathbf{r} \]

The wheel’s forward direction is expressed by the unit vector \( \mathbf{v} \)

Note that \( \mathbf{r} \) points orthogonally to the line from \( P \) to the wheel (why?); \( \mathbf{t} \) could point in any direction.
Finally, the “component of motion in the wheel direction” is obtained...
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Finally, the “component of motion in the wheel direction” is obtained... by taking the dot product of $\dot{w}_R = t + r$ with $v$...

COVERED ON BOARD

Requires the trigonometric identities:

1. \[
\sin(x + y) = \sin x \cos y + \cos x \sin y \]
2. \[
\cos(x + y) = \cos x \cos y - \sin x \sin y \]


Finally, the “component of motion in the wheel direction” is obtained... by taking the dot product of $\dot{w}_R = t + r$ with $v$...

**COVERED ON BOARD**

Requires the trigonometric identities:

$$\sin(x + y) = \sin x \cos y + \cos x \sin y$$
Finally, the “component of motion in the wheel direction” is obtained... by taking the dot product of $\mathbf{w}_R = t + r$ with $\mathbf{v}$...

**COVERED ON BOARD**

Requires the trigonometric identities:

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\begin{align*}
\sin(x + y) &= \sin x \cos y + \cos x \sin y \\
\cos(x + y) &= \cos x \cos y - \sin x \sin y
\end{align*}
\]
Finally, the “component of motion in the wheel direction” is obtained... by taking the dot product of $\dot{w}_R = t + r$ with $v$...

COVERED ON BOARD

Requires the trigonometric identities:

\[
\sin(x + y) = \sin x \cos y + \cos x \sin y \\
\cos(x + y) = \cos x \cos y - \sin x \sin y
\]
The Sliding Constraint

- We now wish to express the following more formally,

\[ \text{Component of motion orthogonal to wheel direction} = 0 \]

The derivation is quite similar to that of the rolling constraint, with the following change:

For the rolling constraint we had

\[ v \cdot (r + t) = r \dot{\phi} \]

Replace \( v \) with \( n \), orthogonal to \( v \); Replace \( r \dot{\phi} \) with 0

Expanding \( n \cdot (r + t) = 0 \) we arrive at the sliding constraint in its final form

\[ \cos(\alpha + \beta) \sin(\alpha + \beta) l \sin \beta R(\theta) \dot{\xi} = 0 \]
We now wish to express the following more formally,

- Component of motion orthogonal to wheel direction = 0

The derivation is quite similar to that of the rolling constraint, with the following change:

For the rolling constraint we had

\[ v \cdot (r + t) = r \dot{\varphi} \]

Replace \( v \) with \( n \), orthogonal to \( v \); Replace \( r \dot{\varphi} \) with 0

Expanding \( n \cdot (r + t) = 0 \) we arrive at the sliding constraint in its final form

\[ \cos(\alpha + \beta) \sin(\alpha + \beta) l \sin \beta \]

\[ R(\theta) \dot{\xi} = 0 \]
We now wish to express the following more formally,

- Component of motion orthogonal to wheel direction $= 0$

The derivation is quite similar to that of the rolling constraint, with the following change:

For the rolling constraint, we had $v \cdot (r + t) = r \dot{\phi}$

Replace $v$ with $n$, orthogonal to $v$; Replace $r \dot{\phi}$ with 0

Expanding $n \cdot (r + t) = 0$ we arrive at the sliding constraint in its final form:

$$\begin{bmatrix} \cos(\alpha + \beta) & \sin(\alpha + \beta) \\ \sin \beta \end{bmatrix} R(\theta) \dot{\xi}_I = 0$$
We now wish to express the following more formally,
- Component of motion orthogonal to wheel direction = 0
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We now wish to express the following more formally,
- Component of motion orthogonal to wheel direction = 0

The derivation is quite similar to that of the rolling constraint, with the following change:
- For the rolling constraint we had \( \mathbf{v} \cdot (\mathbf{r} + \mathbf{t}) = r\dot{\phi} \)

Expanding \( \mathbf{n} \cdot (\mathbf{r} + \mathbf{t}) = 0 \) we arrive at the sliding constraint in its final form:

\[
\begin{align*}
\cos(\alpha + \beta) \sin(\alpha + \beta) l \sin\beta R(\theta) \dot{\xi} = 0
\end{align*}
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The Sliding Constraint

- We now wish to express the following more formally,
  - Component of motion orthogonal to wheel direction $= 0$
- The derivation is quite similar to that of the rolling constraint, with the following change:
  - For the rolling constraint we had $\mathbf{v} \cdot (\mathbf{r} + \mathbf{t}) = r \dot{\phi}$
  - Replace $\mathbf{v}$ with $\mathbf{n}$, orthogonal to $\mathbf{v}$
We now wish to express the following more formally,
- Component of motion orthogonal to wheel direction = 0

The derivation is quite similar to that of the rolling constraint, with the following change:
- For the rolling constraint we had \( \mathbf{v} \cdot (\mathbf{r} + \mathbf{t}) = r\phi \)
- Replace \( \mathbf{v} \) with \( \mathbf{n} \), orthogonal to \( \mathbf{v} \)
We now wish to express the following more formally,

- Component of motion orthogonal to wheel direction = 0

The derivation is quite similar to that of the rolling constraint, with the following change:

- For the rolling constraint we had \( \mathbf{v} \cdot (\mathbf{r} + \mathbf{t}) = r\dot{\phi} \)
- Replace \( \mathbf{v} \) with \( \mathbf{n} \), orthogonal to \( \mathbf{v} \); Replace \( r\dot{\phi} \) with 0
We now wish to express the following more formally,
- Component of motion orthogonal to wheel direction = 0

The derivation is quite similar to that of the rolling constraint, with the following change:
- For the rolling constraint we had \( \mathbf{v} \cdot (\mathbf{r} + \mathbf{t}) = r \dot{\phi} \)
- Replace \( \mathbf{v} \) with \( \mathbf{n} \), orthogonal to \( \mathbf{v} \); Replace \( r \dot{\phi} \) with 0

Expanding \( \mathbf{n} \cdot (\mathbf{r} + \mathbf{t}) = 0 \) we arrive at the sliding constraint in its final form
\[
\cos(\alpha + \beta) \sin(\alpha + \beta) l \sin \beta \mathbf{R}(\theta) \dot{\xi} \mathbf{I} = 0
\]
We now wish to express the following more formally,

- Component of motion orthogonal to wheel direction = 0

The derivation is quite similar to that of the rolling constraint, with the following change:

- For the rolling constraint we had $\mathbf{v} \cdot (\mathbf{r} + \mathbf{t}) = r\dot{\phi}$
- Replace $\mathbf{v}$ with $\mathbf{n}$, orthogonal to $\mathbf{v}$; Replace $r\dot{\phi}$ with 0

Expanding $\mathbf{n} \cdot (\mathbf{r} + \mathbf{t}) = 0$ we arrive at the sliding constraint in its final form
We now wish to express the following more formally,

- Component of motion orthogonal to wheel direction = 0

The derivation is quite similar to that of the rolling constraint, with the following change:

- For the rolling constraint we had \( \mathbf{v} \cdot (\mathbf{r} + \mathbf{t}) = r \dot{\phi} \)
- Replace \( \mathbf{v} \) with \( \mathbf{n} \), orthogonal to \( \mathbf{v} \); Replace \( r \dot{\phi} \) with 0

Expanding \( \mathbf{n} \cdot (\mathbf{r} + \mathbf{t}) = 0 \) we arrive at the sliding constraint in its final form

\[
[\cos(\alpha + \beta) \sin(\alpha + \beta) \ l \sin \beta] \ R(\theta) \hat{\xi}_1 = 0
\]