Supplementary Material: The Rolling and Sliding Constraints

Computer Science 4766/6912

Department of Computer Science
Memorial University of Newfoundland

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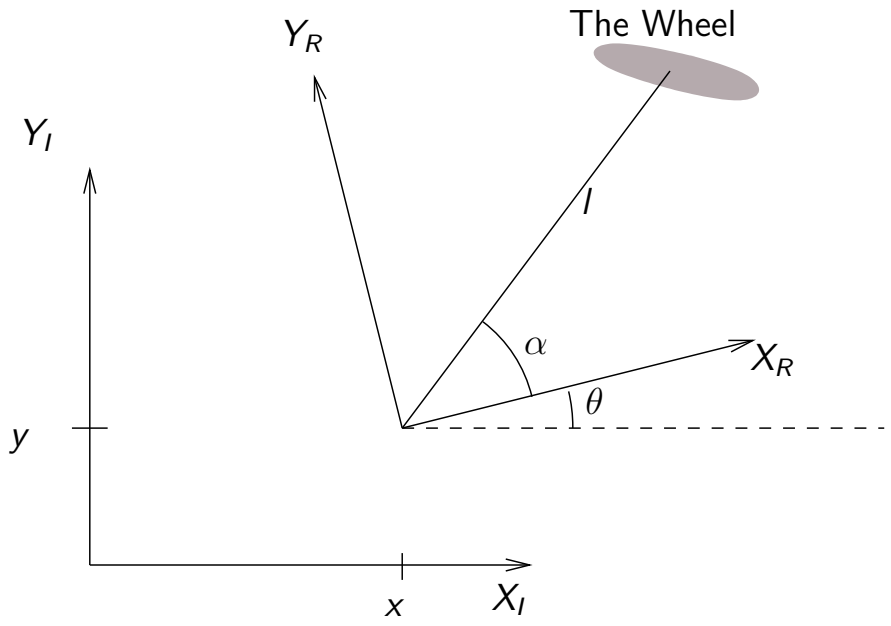

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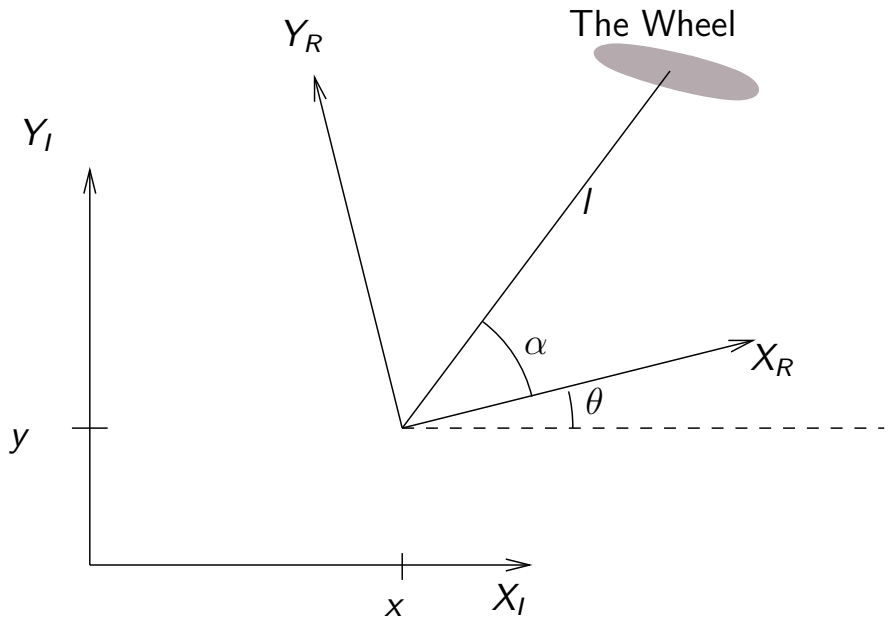

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where $\boldsymbol{t} = [\dot{x_R}, \dot{y_R}]^T$ represents the motion of the wheel due to **translation** of the robot and $\boldsymbol{r} = l\dot{\theta}[-\sin\alpha, \cos\alpha]^T$ represents the motion due to the robot's **rotation**

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\dot{\boldsymbol{w}}_{\boldsymbol{R}} &= \begin{bmatrix} \dot{x}_R \\ \dot{y}_R \end{bmatrix} + l\dot{\theta} \begin{bmatrix} -\sin\alpha \\ \cos\alpha \end{bmatrix} \\
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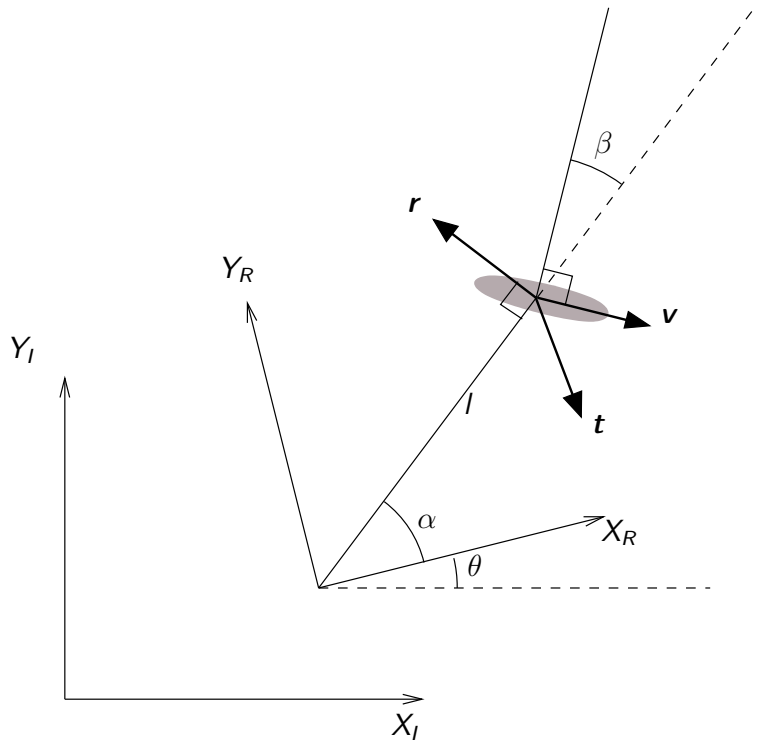

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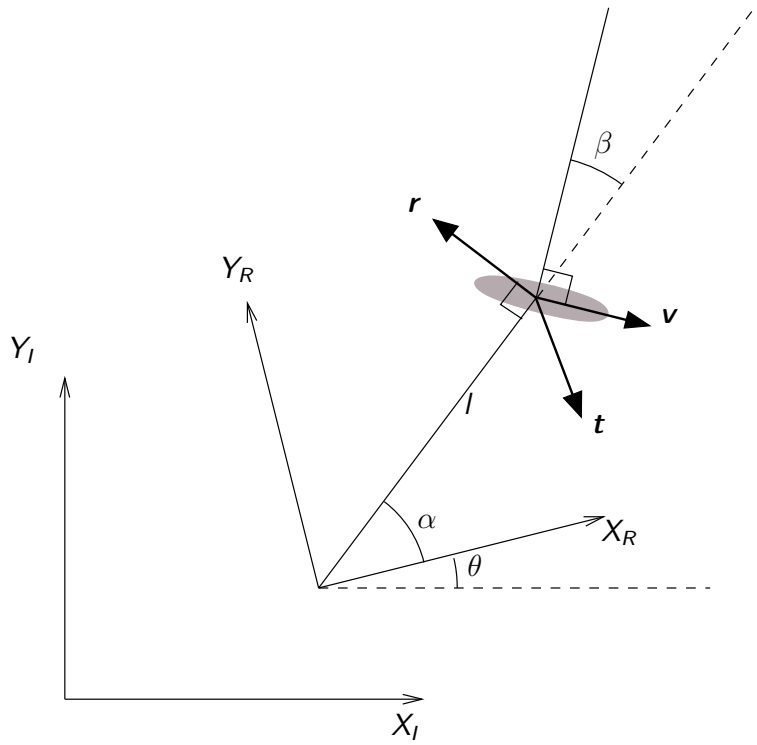

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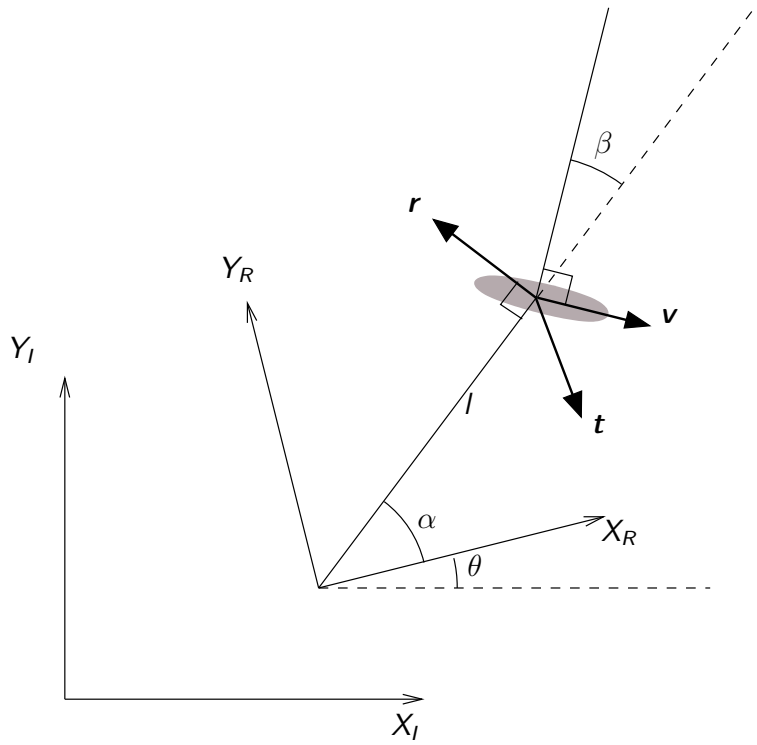

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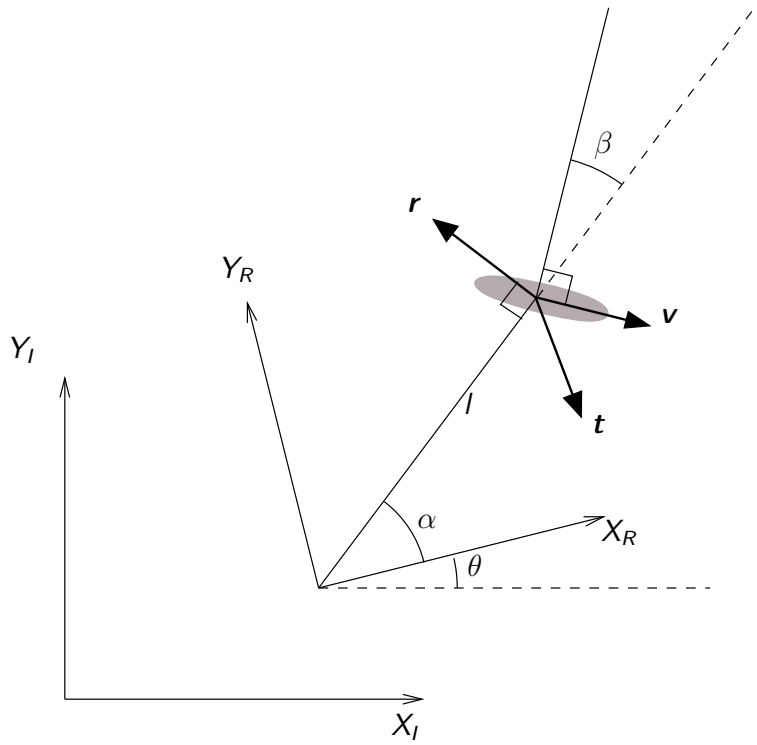

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$$[\cos(\alpha + \beta) \;\; \sin(\alpha + \beta) \;\; l \sin \beta] \; \boldsymbol{R}(\theta)\dot{\boldsymbol{\xi}}_{\boldsymbol{I}} = 0$$