Unit 5: Stability

Engineering 5821:
Control Systems I

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1 Stability

1 Routh-Hurwitz Criterion

- Special Case: Zero in First Column
- Special Case: Row of Zeros
- Stability Design Example
System stability can be defined w.r.t. the stability of the natural response, or of the total response (natural + forced).

**Stability of the natural response:**

- If the natural response approaches 0 as $t \to \infty$ the system is **stable**. If the response approaches $\infty$ the system is **unstable**.
- If the natural response remains constant or oscillates the system is **marginally stable**.

**Stability of the total response.** Sometimes referred to as bounded-input, bounded-output (BIBO) stability:

- If *every* bounded input yields a bounded output, the system is **stable**.
- If *any* bounded input yields an unbounded output, the system is **unstable**.
Considering the natural response, the system is stable as long as all closed-loop system poles lie only in the left half-plane (LHP):

Poles in the right half-plane (RHP) yield exponential growth—hence **instability**.
Pairs of poles on the imaginary axis lead to undamped sinusoids:

\[ A \cos(\omega t + \phi) \]

The presence of such poles make the natural response (at best) **marginally stable**. If a pair of poles is repeated we obtain responses of the form:

\[ At^n \cos(\omega t + \phi) \]

where \( n = 1 \) for the second repeated pair, 2 for the third \ldots Thus, if poles on the imaginary axis are repeated the system is **unstable**.
(a) Stable system

\[ R(s) = \frac{1}{s} \quad + \quad E(s) \quad \rightarrow \quad C(s) \]

\[ \frac{3}{s(s+1)(s+2)} \]

Stable system's closed-loop poles (not to scale)

(b) Unstable system

\[ R(s) = \frac{1}{s} \quad + \quad E(s) \quad \rightarrow \quad C(s) \]

\[ \frac{7}{s(s+1)(s+2)} \]

Unstable system's closed-loop poles (not to scale)
System stability is determined entirely by the location of the roots of the closed-loop transfer function’s denominator. For the example above this polynomial was,

\[ s^3 + 3s^2 + 2s + k \]

where \( k \) was either 3 or 7. With \( k = 3 \) the system was stable, but became unstable for \( k = 7 \). The roots of the denominator polynomial govern the system’s stability. These roots are located wherever the following characteristic equation holds,

\[ s^3 + 3s^2 + 2s + k = 0 \]
If the denominator polynomial has either of the following characteristics, then it will have RHP roots and the system will be unstable:

- **If the polynomial coefficients differ in sign (i.e. not all positive or all negative).** We have stability iff the polynomial can be factored as follows where the real part of each root $r_i$ is positive:

  $$(s + r_1)(s + r_2) \cdots (s + r_n).$$

  This will always generate a polynomial with all positive coefficients (or all negative if we multiply everything by -1). So differently signed coefficients indicate instability.

- **If the polynomial has missing coefficients.**

However, there are unstable polynomials with all coefficients $\geq 0$.

One option is to determine the locations of the roots with Matlab. However, we will need to establish analytical criteria for stability...
The **Routh-Hurwitz criterion** allows us to obtain a *count* of the number of poles in the LHP, RHP, and on the $j\omega$-axis. It does not tell us where the poles are, only how many there are of each category.

We must first generate a Routh table. Assume our system is as follows (we care only about the denominator):

$$R(s) \quad N(s) \quad C(s)$$

$$a_4 s^4 + a_3 s^3 + a_2 s^2 + a_1 s + a_0$$

Each row of the Routh table is labelled from the highest denominator power down to $s^0$. Then start with the highest power and list every other coefficient in the first row of the table. In the second row list the skipped coefficients in order.

\[
\begin{array}{cccc}
  s^4 & a_4 & a_2 & a_0 \\
  s^3 & a_3 & a_1 & 0 \\
  s^2 & & & \\
  s^1 & & & \\
  s^0 & & & \\
\end{array}
\]
We continue by filling in each entry as a negative determinant divided by the first column of the row above. The left column of the determinant comes from the first entries of the two rows above. The right column comes from the elements of columns above and to the right.

\[
\begin{align*}
\frac{s^4}{s^3} & = \frac{a_4}{a_3} \quad \frac{a_2}{a_1} \quad \frac{a_0}{0} \\
\frac{s^2}{s^1} & = \frac{a_3}{a_1} \quad \frac{a_3}{b_1} \quad \frac{a_3}{b_1} \\
\frac{s^0}{s^0} & = \frac{b_1}{b_1} \quad \frac{b_1}{b_1} \quad \frac{b_1}{b_1} \\
\end{align*}
\]
e.g. \[ G(s) = \frac{1000}{s^3 + 10s^2 + 31s + 1030} \]

\[
\begin{array}{cc}
\hline
s^3 & 1 \\
\hline
s^2 & 10 \\
& 1 \\
\hline
s^1 & 1 \\
& 103 \\
\hline
s^0 & -72 \\
& 0 \\
\hline
\end{array}
\]

\[
\begin{array}{cc}
\hline
& 31 \\
1 & 1030 \\
\hline
1 & 103 \\
\hline
\end{array}
\]

\[
\begin{array}{cc}
\hline
1 & 0 \\
\hline
1 & 0 \\
\hline
\end{array}
\]

\[
\begin{array}{cc}
\hline
1 & 0 \\
\hline
1 & 0 \\
\hline
\end{array}
\]

Note: For convenience, we can multiply any row by a positive constant. This was done in the \( s^2 \) row above.

**Routh Table Interpretation:** The number of RHP roots equals the number of sign changes in the first column. (*The proof of this statement is beyond the scope of this course*)

For the example above there are two sign changes. Therefore the system has two RHP poles and is **unstable**.
Note that the width of the table is governed by the formula $\lceil \frac{n+1}{2} \rceil$ where $n$ is the order of the polynomial and $n + 1$ is the number of coefficients.

e.g. $7s^7 + 6s^6 + 5s^5 + 4s^4 + 3s^3 + 2s^2 + s + k$ requires 4 columns.
e.g. $8s^8 + 7s^7 + 6s^6 + 5s^5 + 4s^4 + 3s^3 + 2s^2 + s + k$ requires 5 columns.

You can forget this width rule as long as you remember to stop filling in entries when a column of zeros is obtained.
If the first column of any row becomes 0 we have to handle this as a special case. Replace the 0 with a small positive constant $\epsilon$. Continue as normal...

e.g. $s^5 + 2s^4 + 3s^3 + 6s^2 + 5s + 3$

\[
\begin{array}{cccc}
s^5 & 1 & 3 & 5 \\
s^4 & 2 & 6 & 3 \\
s^3 & \epsilon & \frac{7}{2} & 0 \\
s^2 & \frac{6\epsilon-7}{\epsilon} & 3 & 0 \\
s^1 & \frac{42\epsilon-49-6\epsilon^2}{12\epsilon-14} & 0 & 0 \\
s^0 & 3 & 0 & 0
\end{array}
\]

Signs: $+, +, +, -, +, +$

Thus, there are two sign changes and therefore two RHP poles. The system is unstable.
A row of zeros (ROZ) will occur if there is an even polynomial (EP) that is a factor of the polynomial we are working on. The coefficients of this EP are found in the row above the zero row. The power of the EP is given by the row above the zero row. We continue the Routh table by replacing the zero row with the coefficients of the derivative of the EP.

e.g. $s^5 + 7s^4 + 6s^3 + 42s^2 + 8s + 56$
e.g. \( s^5 + 7s^4 + 6s^3 + 42s^2 + 8s + 56 \)

\[
\begin{array}{ccc}
\text{s}^5 & 1 & 6 & 8 \\
\text{s}^4 & 7 & 42 & 56 \quad \text{Multiply by } 1/7 \\
1 & 6 & 8 \\
\text{s}^3 & 0 & 0 & 0 \quad \text{ROZ! Form } EP = s^4 + 6s^2 + 8 \text{ from above row's coefficients. Differentiate to obtain } 4s^3 + 12s. \\
\end{array}
\]

\[
\begin{array}{ccc}
4 & 12 & 0 \quad \text{Multiply by } 1/4 \\
1 & 3 & 0 \\
3 & 8 & 0 \\
1/3 & 0 & 0 \\
8 & 0 & 0 \\
\end{array}
\]

Interpretation: We must split our interpretation and consider first the part above the ROZ separately from the part below. Everything below the ROZ is a test of EP. In this case we see no sign changes above or below. Therefore there are no RHP poles. Is the system stable or only marginally stable?
In the previous example the EP $s^4 + 6s^2 + 8$ was a factor of the original polynomial. EP’s have the property that their roots are always symmetric about the origin. These roots can be purely real, purely imaginary, or complex, but each root will have a match located $180^\circ$ opposite.

This means that a system with an even polynomial factor will either be unstable (A or C) or marginally stable.

In our example $s^4 + 6s^2 + 8$ has four purely imaginary roots (two pairs of roots as in B). Therefore the system is marginally stable.
e.g. \( s^8 + s^7 + 12s^6 + 22s^5 + 39s^4 + 59s^3 + 48s^2 + 38s + 20 \)

| \( s^8 \) | 1 | 12 | 39 | 48 | 20 |
| \( s^7 \) | 1 | 22 | 59 | 38 | 0  |
| \( s^6 \) | -10 | -20 | 10 | 20 | 0  | Multiply by 1/10 |
|          | -1 | -2  | 1  | 2  | 0  |
| \( s^5 \) | 20 | 60  | 40 | 0  | 0  | Multiply by 1/20 |
|          | 1  | 3   | 2  | 0  | 0  |
| \( s^4 \) | 1  | 3   | 2  | 0  | 0  |
| \( s^3 \) | 0  | 0   | 0  | 0  | 0  | ROZ! EP = \( s^4 + 3s^2 + 2 \) |

|        | 4 | 6 | 0 | 0 | 0 | 0 | Multiply by 1/4 |
|        | 2 | 3 | 0 | 0 | 0 | 0 | Multiply by 1/2 |
| \( s^2 \) | 3/2 | 2 | 0 | 0 | 0 | 0 | Multiply by 2 |
|          | 3 | 4 | 0 | 0 | 0 | 0 |
| \( s^1 \) | 1/3 | 0 | 0 | 0 | 0 | 0 |
| \( s^0 \) | 4 | 0 | 0 | 0 | 0 | 0 |

Interpretation: Two sign changes above the line, implies two RHP poles. The system is unstable, but we may still want to know where the poles are. Below the line there are no sign changes. Since this part comes from the EP which has symmetrical roots, we know that all four of its roots must be on the \( j\omega \)-axis. The final two poles belong to the original polynomial and must be in the LHP.
e.g. Find the range of gain $K$ for the closed-loop transfer function $T(s)$ that will lead to stable, unstable, and marginally stable behaviour. Assume $K > 0$.

$$T(s) = \frac{K}{s^3 + 18s^2 + 77s + K}$$

$\begin{array}{ccc}
s^3 & 1 & 77 \\
s^2 & 18 & K \\
s^1 & \frac{1386-K}{18} & 0 \\
s^0 & K & 0 \end{array}$

If $K < 1386$ there are no sign changes and the system is stable.

If $K > 1386$ we have two sign changes and the system is unstable.

If $K = 1386$ we have a ROZ. The EP is $18s^2 + 1386$. Differentiate to obtain $36s$. Replacing the ROZ with $[36, 0]$ the next row becomes $[1386, 0]$. There are no sign changes so the EP must have both roots on the $j\omega$ axis. Thus, the system is marginally stable.