Rotational Mechanical Systems

Unit 2: Modeling in the Frequency Domain
Part 6: Modeling Rotational Mechanical Systems

Engineering 5821:
Control Systems I

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January 22, 2010

Rotational mechanical systems are modelled in almost the same way as translational systems except that:

- We replace displacement, \( x(t) \) with angular displacement \( \theta(t) \); Angular velocity is \( \omega(t) \)
- We replace force with torque

For a force \( F \) acting on a body at point \( P \), torque is defined as,

\[ T = FR \sin \phi \]

where \( R \) is the distance from \( P \) to the body’s axis of rotation and \( \phi \) is the angle the force makes to the ray from the axis of rotation to \( P \). Hence, if the force is perpendicular to the axis of rotation then,

\[ T = FR \]

A rotating body can be considered a system of particles with masses \( m_1, m_2, m_3, \ldots \). The **moment of inertia** is defined as,

\[ J = m_1R_1^2 + m_2R_2^2 + m_3R_3^2 + \cdots \]

The total kinetic energy is,

\[ K = \frac{1}{2}J\omega^2 \]

Recall that the kinetic energy for a translational system is \( \frac{1}{2}mv^2 \).

So \( J \) is analogous to mass in translational motion. Also, similar to the equation \( F = ma \) in translational systems, we can relate torque and angular acceleration,

\[ T(t) = J\frac{d\omega}{dt} = J\frac{d^2\theta}{dt^2} \]

We define the components of our rotational system as springs, viscous dampers, and rotating masses.

<table>
<thead>
<tr>
<th>Component</th>
<th>Torque-angular velocity</th>
<th>Torque-angular displacement</th>
<th>Impedance ( Z_M(s) = T(t)\theta(t) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Spring</td>
<td>( T(t) = K \int_0^t \omega(t) , dt )</td>
<td>( T(t) = K\theta(t) )</td>
<td>( K )</td>
</tr>
<tr>
<td>Viscous damper</td>
<td>( T(t) = D\theta(t) )</td>
<td>( T(t) = D\frac{d\theta(t)}{dt} )</td>
<td>( D )</td>
</tr>
<tr>
<td>Inertia</td>
<td>( T(t) = J\frac{d\theta(t)}{dt} )</td>
<td>( T(t) = J\frac{d^2\theta(t)}{dt^2} )</td>
<td>( J I_2 )</td>
</tr>
</tbody>
</table>

Note: The following set of symbols and units is used throughout this book: \( T(t) \) = N-m (newton-meters), \( \theta(t) \) = rad (radians), \( \omega(t) \) = rad/s (radians/second), \( K \) = N-m/rad (newton-meters/radian), \( D \) = N-m/rad (newton-meters/second/radian), \( J \) = kg-m^2 (kilograms-meters^2 = newton-meters/second^2/radian).
e.g. Find the transfer function \( \frac{\theta_2(s)}{T(s)} \) for the following system:

We model the system in (a) as consisting of two sections joined by a spring. We draw a free-body diagram of \( J_1 \):

(a) \( J_1 \) rotating, \( J_2 \) held still (b) \( J_2 \) rotating, \( J_1 \) held still (c) All torques on \( J_1 \)

Sum the torques in (c)

\[-K\theta_1(s) + (J_2s^2 + D_2s + K)\theta_2(s) = 0\]

We can easily solve these two linear equations for the transfer function \( \frac{\theta_2(s)}{T(s)} \).

Rarely do we see mechanical systems without gear trains. Gears allow us to trade-off speed for torque.

We will assume that connected gears fit perfectly together. However, in reality gears exhibit backlash where one gear will move through a small angle before its teeth meet those of the other gear. This is a non-linear effect that we will not model analytically.
The input gear on the left has radius $r_1$ and $N_1$ teeth. It is rotated by $\theta_1(t)$ due to a torque $T_1(t)$. What is the relationship between the rotation of the input gear and that of the output gear, $\theta_2(t)$?

Although, the angles will differ, the arc length through which both gears turn will be the same:

$$r_1 \theta_1 = r_2 \theta_2$$

Therefore the relation between angles is as follows,

$$\theta_2 = \frac{r_1}{r_2} \theta_1$$

Since the number of teeth is proportional to the radius, then the following also holds,

$$\theta_2 = \frac{N_1}{N_2} \theta_1$$

We can relate $T_1$ and $T_2$ through energy considerations. The amount of work done by the rotation of gear 1 is $T_1 \theta_1$. We are assuming that no energy is lost, therefore

$$T_1 \theta_1 = T_2 \theta_2 \implies \frac{T_2}{T_1} = \frac{N_2}{N_1}$$

The relationships between gears are pictured as transfer functions below:

Now assume we are interested in the relation between $T_1$ and $\theta_1$. The torque due to each of the impedances on shaft 2 can be reflected to an equivalent torque on shaft 1. Consider the torque due to the damper on shaft 2:

$$T_{D2} = Ds \theta_2(s) = Ds \frac{N_1}{N_2} \theta_1(s)$$

The relationship between the torques due to the damper is,

$$T_{D1} = \frac{N_1}{N_2} T_{D2} \implies T_{D1} = \frac{N_1}{N_2} Ds \frac{N_1}{N_2} \theta_1(s) = \left( \frac{N_1}{N_2} \right)^2 Ds \theta_1(s)$$
The general pattern for the reflectance of impedances is as follows:

\[ T_{\text{dest}} = \left( \frac{N_{\text{dest}}}{N_{\text{src}}} \right)^2 Z M \theta_{\text{dest}} \]

In this manner we can reflect all impedances on shaft 2 to shaft 1:

This system can be modelled as follows,

\[ \left( J s^2 \left( \frac{N_1}{N_2} \right)^2 + D s \left( \frac{N_1}{N_2} \right)^2 + K \left( \frac{N_1}{N_2} \right)^2 \right) \theta_1(s) = T_1(s) \]

E.g., find the transfer function \( \theta_2(s)/T_1(s) \) for the following system,

Since the output is defined as \( \theta_2(s) \) we should reflect the impedances from shaft 1 onto shaft 2:

We can now write the equation of motion:

\[ (J_e + D_e + K_2) \theta_2(s) = T_1(s) \frac{N_2}{N_1} \]

where,

\[ J_e = J_1 \left( \frac{N_2}{N_1} \right)^2 + J_2 \]
\[ D_e = D_1 \left( \frac{N_2}{N_1} \right)^2 + D_2 \]

Therefore

\[ G(s) = \frac{\theta_2(s)}{T_1(s)} = \frac{N_2 / N_1}{J_e s^2 + D_e s + K_2} \]
If we allow our gears to be large enough we can obtain any desired gear ratio. However, it is usually impractical to allow gears with large radii. Instead, **gear trains** are employed.

The equivalent gear ratio is the product of gear ratios for pairs of meshed gears.

e.g. Reflect all impedances in the abbreviated schematic below onto the input shaft:

![Diagram](image)

\[ \theta_1 = \frac{N_1}{N_2} \frac{N_3}{N_4} \theta_1 \]

Solution:

Notice that we are assuming the rotations of all other shafts are directly tied through the gear train to \( \theta_1 \).