Unit 2: Modeling in the Frequency Domain
Part 1: Complex Frequency

Engineering 5821:
Control Systems I

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January 7, 2010
1 Complex Frequency

1 Complex Representation of Time-Domain Signals

1 Example

1 The s-plane
Consider a typical electrical system: the series RL circuit excited by a DC source. We can develop a DE for this system:

\[ L \frac{di(t)}{dt} + Ri(t) = V_s \]

The solution for this equation is of the following form:

\[ i(t) = A + Be^{\alpha t} \]

Where \( A \), \( B \), and \( \alpha \) are constants determined from the circuit’s initial conditions and the DE itself.

Note that the response has two components:

- Forced response: \( A \)
- Natural response: \( Be^{\alpha t} \)
What if the input was an AC source? The forced response would be a sinusoid with the same frequency as the input ($\omega$) only it would differ in amplitude and phase:

$$i(t) = A \cos(\omega t + \theta) + Be^{\alpha t}$$

Either way, the response consists of constants, exponentials, and sinusoids. We will return to this point shortly...
Assume we have some time varying quantity called $x$. We say that $x$ has a complex frequency $s$ when it can be expressed as follows:

$$x(t) = \Re \{X e^{st}\}$$

Where $X$ and $s$ are complex numbers and,

$$s = \sigma + j\omega$$

and,

$$X = A + jB \quad \text{or} \quad Ce^{j\theta}$$

Notice that $s$ must have units of inverse seconds. Consider the signals that can be expressed when $s$ has different values:

$s = 0$

$$x(t) = \Re \{X e^{0t}\}$$

$$= \Re \{A + jB\} = A$$

A constant (i.e. a DC quantity) is represented.
\[ s = \sigma \]

\[
\begin{align*}
\mathbf{x}(t) &= \Re\{Xe^{\sigma t}\} \\
&= \Re\{X\}e^{\sigma t} \\
&= Ae^{\sigma t}
\end{align*}
\]

Notice that if \( \sigma \) is negative we have a decaying exponential. If its positive we have a growing exponential (generally a bad thing to have in a control system).
Returning to our example, we can utilize various techniques to arrive at the constants $A$, $B$, and $\alpha$. The final solution is:

$$i(t) = \frac{V_s}{R} \left(1 - e^{-\frac{R}{L}t}\right)$$

If $V_s = 10$, $R = 5\Omega$, $L = 2H$ then we have,

$$i(t) = 2 - 2e^{-2.5t}$$

This signal is composed of two components with complex frequencies:

$$i(t) = \Re\{2e^{0t}\} - \Re\{2e^{-2.5t}\}$$

The values of $s$ are 0 and -2.5.
Plot of $2 \left(1 - e^{-2.5t}\right)$:
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\[ s = j\omega \]

\[ x(t) = \Re\{Xe^{j\omega t}\} \]
\[ = \Re\{Ce^{j\theta}e^{j\omega t}\} \]
\[ = \Re\{Ce^{i(\omega t+\theta)}\} \]

We now utilize Euler’s identity:

\[ e^{j\theta} = \cos \theta + j \sin \theta \]

Continuing the derivation,

\[ x(t) = \Re\{C \cos(\omega t + \theta) + jC \sin(\omega t + \theta)\} \]
\[ = C \cos(\omega t + \theta) \]

So far these signals with complex frequency \( s \) can represent constants, exponentials, and sinusoids. Thus, the complex frequency representation covers all types of responses from our RL circuit.
**General interpretation:** \( s = \sigma + j\omega \)

\[
x(t) = \Re\{Xe^{(\sigma+j\omega)t}\} \\
= \Re\{Xe^{\sigma t}e^{j\omega t}\} \\
= e^{\sigma t}\Re\{Xe^{j\omega t}\}
\]

On the previous slide we derived \( \Re\{Xe^{j\omega t}\} = C \cos(\omega t + \theta) \). Therefore,

\[
x(t) = Ce^{\sigma t} \cos(\omega t + \theta)
\]

This represents a sinusoid with a decaying or growing envelope.
e.g. $e^{-0.4t} \cos(3x), \ s = -0.4 + j3$

e.g. $e^{0.2t} \cos(4x), \ s = 0.2 + j4$
Notice that $e^{-0.4t} \cos(3x)$ can be represented by $s = -0.4 \pm j3$.

Why? Since $\cos$ is an even function the sign of $\omega$ makes no difference.

Another reason... Let $z$ be a complex number.

\[
\Re\{z\} = \frac{1}{2}(z + z^*)
\]

where $z^* = \Re\{z\} - \Im\{z\}$. Thus we can express a damped or growing sinusoid as half the sum of complex conjugates.

\[
x(t) = Ce^{\sigma t} \cos(\omega t)
\]

\[
x(t) = \Re\{Xe^{(\sigma+j\omega)t}\}
\]

\[
= \frac{X}{2}e^{(\sigma+j\omega)t} + \frac{X}{2}e^{(\sigma-j\omega)t}
\]

If we applied Euler’s identity we would see that the complex parts cancelled out. No matter the sign of $\omega$ we get the same result.
Example

e.g. What is the complex frequency for $v(t)$?

$$v(t) = 100e^{20t} \sin(400\pi t + 75^\circ)$$

The two complex frequencies that both correspond to this signal are $s = 20 + j400\pi$ and $s = 20 - j400\pi$.

e.g. Express $v(t)$ in the form $\Re\{Xe^{st}\}$.

Since $\cos(x - 90^\circ) = \sin(x)$,

$$v(t) = \Re\{100e^{j(-15^\circ)}e^{(20\pm j400\pi)t}\}$$

However, the angle in the exponent really should be in radians:

$$v(t) = \Re\{100e^{-0.261j}e^{(20\pm j400\pi)t}\}$$

Would the complex frequency change if $v(t)$ was expressed with $\cos$ as opposed to $\sin$?
The $s$-plane represents a complex frequency $s$ as a point in the complex plane. For different values of $s$ we get different time-domain responses.