CS 2742 (Logic in Computer Science)
Lecture 6

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1.1 More about implications

In the previous lecture we had the following knights-and-knaves puzzle:

A said: "If I am a knight I'll eat my hat!". Show that A will eat his hat.

Note that this statement is an implication. Lets set p: "A is a knight" and q: "A will eat his hat". Then what A said is $p \to q$. Now, consider the truth table for the statement saying that what A said is truth if and only if A is a knight (that is, $(p \to q) \leftrightarrow p$.)

p	q	$p \to q$	$(p \to q) \leftrightarrow p$
T	T	T	T
T	F	F	F
F	T	T	F
F	F	T	F

So the only situation which is possible (that is, A is a knight and he told the truth or A is a knave and he lied) is when A is a knight and he told the truth. And since what he said is true, and the left hand side of the implication (that is, p) is true, q also has to be true. So A will eat his hat.

Let us look what happens when A is a knave. Then what he said must be false. But the only time $p \to q$ is false is when p is true and q is false (that is, $\neg(p \to q) \iff (p \wedge \neg q)$, you can check by the definition of implication and the DeMorgan's law that this holds). So here is where the contradiction comes: the only time the implication could be false (that is, uttered by a knave) is when its left-hand-side is true (that is, A is a knight).

One of the main things to remember about the implication is that **falsity can imply anything!**. That is, if pigs can fly, then 2+2=5, and also if pigs can fly then 2+2=4.

Both of these are true implications, provided that pigs cannot fly. No matter what kind of statement is q, the implication $F \to q$ is always true.

A brilliant example that shows that falsity can imply anything via a valid argument was presented by the famous logician Bertrand Russell (author of the "Russell's paradox" that we will see later in the course.)

Example 1. Bertrand Russell: "If 2+2=5, then I am the Pope".[1]

Proof. If 2+2=5 then 1=2 by subtracting 3 from both sides.
Bertrand Russell and Pope are two people.
Since 1=2, Bertrand Russell and Pope are one person.

$\square$

Note that the steps of this proof are perfectly fine logically. Every line follows from the previous line correctly. The strange conclusion follows instead from 2+2=5 being a false statement.

1.2 Valid and invalid arguments

Bertran Russel's proof of "if 2+2=5, then I am the Pope" is an example of a valid argument (that is, every step logically followed from the previous). In this subsection we will discuss what constitutes a valid argument, and later discuss methods for proving that arguments are valid.

We will start with a (propositional form of) Aristotle's classic example of a valid argument (his original version had a quantifier; we will get to that in a few lectures).

> If Socrates is a man, then Socrates is mortal
> Socrates is a man
> $\therefore$ Socrates is mortal

Terminology: the final statement is called *conclusion*, the rest are *premises*. The symbol $\therefore$ reads as *therefore*. An argument is *valid* if no matter what statements are substituted into variables, if all premises are true then the conclusion is true. That is, if there are formulas $P_1, \ldots, P_k$ that are all the premises (one per line, as above), and a formula C is the conclusion, then an argument is valid iff $P_1 \wedge P_2 \wedge \cdots \wedge P_k \to C$ is a tautology.

[1] According to other sources, the statement Bertrand Russell was proving was "if 1+1=1, then I am the Pope". In this case, the first line can be omitted.

The valid form of agument is called *rules of inference*. The most known is called *Modus Ponens* ("method of affirming"). Its contrapositive is called *Modus Tollens* ("method of denying"):

Modus Ponens	*Modus Tollens*
If p then q	If p then q
p	$\neg q$
$\therefore q$	$\therefore \neg p$

This is another way to describe "proof by contrapositive". Similarly we can write the proof by cases, by contradiction, by transitivity and so on. They can be derived from the original logic identities. For example, modus ponens becomes $((p \to q) \wedge p) \to q$.

Example 2. The general form of the proof by cases above can be written as follows:

$p \vee q$	"n is even or n is odd"
$p \to r$	"if n is even then $\lfloor (n+1)/2 \rfloor = \lceil n/2 \rceil$ "
$q \to r$	"if n is odd then $\lfloor (n+1)/2 \rfloor = \lceil n/2 \rceil$ "
$\therefore r$	"therefore $\lfloor (n+1)/2 \rfloor = \lceil n/2 \rceil$ "

To show that an argument is invalid give a truth table where all premises are true and the conclusion is false. That is, show that a propositional formula of the form $\wedge$(of premises) $\to$ conclusion is not a tautology.

Example 3. For example, although modus ponens is valid, the following is not a valid inference:

$p \to q$
q INVALID ARGUMENT!
$\therefore p$

We can write it as $((p \to q) \wedge q) \to p$ and show that this is not a tautology using a truth table.

p	q	$p \to q$	$(p \to q) \wedge q$	$(p \to q) \wedge q \to p$
T	T	T	T	T
T	F	F	F	T
F	T	T	T	**F**
F	F	T	F	T

The truth table showed us a situation when both premises $(p \to q)$ and q are true, but the conclusion p is false. Therefore, $((p \to q) \land q) \to p$ is not a tautology and thus the argument based on it is not a valid argument.

However, note that if any of the premises are false, a valid argument can produce a most weird conclusion: remember that if p is false in $p \to q$, then $p \to q$ is true for any value of q. Thus, if a premise is false, a false conclusion can be reached (even though an argument is valid); a true conclusion can be reached as well.